Complexity and Control of Collective Learning Dynamics in a Simple Model of Market

Massaki Kunigami*, Takao Terano**
Graduate School of Business Science, Tsukuba University*
3-29 Otsuka Bunkyo-ku Tokyo, JAPAN
Computational Intelligence and Systems Science Tokyo Institute of Technology**
4259 Nagatsuda-Cho, Midori-ku, Yokohama 226-8502, Japan
kunigami@gssm.otsuka.tsukuba.ac.jp, terano@dis.titech.ac.jp

Abstract. Although there are different arguments as to whether a real market has chaotic dynamics, this paper presents a new simple but plausible economic market model and investigates the endogenous complex action of the market dynamics. The model focuses on the complexity of "micro-macro interaction" between the market and numberless bounded rational agents, who individually select or change their products among several variations, instead of concerning not each rational players' complex behavior. Collective learning dynamics of this market has equivalent formulas of Replicator Dynamics in the evolitional game theory. The main contribution of the paper is summarized as follows. (1) In spite of deriving from the linear inverse demand function, linear cost function and continuous time modeling, this market dynamics shows a bifurcation with proper chaotic behavior. (2) A chaos control technique can be applied to harness and stabilize the chaotic market behaviors of this dynamics.

1 Introduction

Although there are different arguments as to whether a real market has chaotic dynamics, there exist various complex phenomena in economic systems. This paper presents a new simple but plausible economic market model and investigates the endogenous complex behavior and harnessing of the market dynamics. This problem is a relevant case for analyzing complex economic systems by agent-based dynamical modeling.

In the literature, we find many studies on complex behaviors on Cournot market dynamics for duopoly or oligopoly dynamics. (Puu&Sushko [1]) In Cournot type model (Cournot [2]), one can reduce complexity of consumption and demand into an inverse demand function, so that can make a model of producers' strategic behavior. Then, with nonlinear price/cost structure or learning delay, simple dynamic model can describe complex market behavior in duopoly or oligopoly.

On the other hand, complex system studies are interested in the complex "micro-macro interaction" caused between micro agents and macro market status rather than players' complex strategies. From this viewpoint, this research presents market
dynamics by integrating the Cournot type price mechanism with methodologies of Social Learning Dynamics (SLD) (Deguchi[3]).

Major contributions of this research are as follows. Extend the Cournot type inverse demand function for goods with several variations that affect each other. Assume there are numberless agents who select/change variation to produce. Deguchi[3] proposes that Social Learning Dynamics can describe a model of such collective Markov decision process. Deguchi[3] also shows that SLD has same formula as Replicator Dynamics in the evolutionary game theory. (Hofbauer [4])

We propose a new market dynamics, which describes micro-macro interaction between agents and price-cost structure, without using biological analogy of reproduction and evolution. This continuous dynamics of the market with linear price-cost function shows chaotic motion without using discrete folding pie process derived from non-monotonic reaction curve. As "sheep dog" agents, a chaos control mechanism for continuous system can stabilize the chaotic motion of the market.

2 The Proposed Model

This section derives the new market dynamics by integrating Cournot-type price-cost structure and Social Learning Dynamics. Approaches of modeling are as follows.

(a) Define an inverse demand function and a cost function for several variations of the goods. Supply of each variation has negative or positive effects to the others' price and cost.

(b) Define agents as "bounded rational" numberless producers. They cannot know exact properties of a price curve or others' decision, but can change the variation of goods to produce in a Markov-learning process based on macroscopic information from the market.

2.1 Structure of Price and Market

To generalize Cournot type price mechanism, define a relation between price and production quantity of n variations of the goods. The production vector is \( x(t) = (x_1, x_2, ..., x_n)^T \), the variables \( x_i \) is defined as the quantity of the \( i^{\text{th}} \) variations of goods produced and shipped into the market. (This paper generally uses italic for numbers, bold for vectors and matrices. And upper suffix \( T \) of a vector means its transpose.)

The following formula represents a linear inverse demand relation between \( x \) and the price vector \( p \) of which each element indicates market price of \( i^{\text{th}} \) kind.

\[
p = p(x) = p_0 + Px = (P_0 + P)x, \quad (p(x))_i \geq 0 \quad \forall i,
\]

\[
p_0 \equiv p_0(1,1,\ldots,1)^T \in \mathbb{R}_+^n,
\]

\[
P_0 \equiv (p_0,p_0,\ldots,p_0) \in M_{n\times n}, \quad P \equiv (P_0) \in M_{n\times n}.
\]
Complexity and Control of Collective Learning Dynamics in a Simple Model of Market

Here, each $p_{0i}$ is the p-section of the $i^{th}$ price surface. Suppose all elements of $p_0$ are same, i.e. $p_0 = (p_0, p_0, ..., p_0)$, because all the limit utilities of variation can be regarded as same when the goods are only supplied minimum unit into the market. The matrix $P = \{p_{0i}\}$ indicates mutual price effects between price of $i^{th}$ variation and production quantity of $j^{th}$ variation. To simplify, assume that each agent can produce just one unit of the specific variation of goods, in a unit time. In addition, let the total population of agents be normalized as 1 (i.e. $\Sigma x_j = 1$), so that quantity of each kind of goods is identified with corresponding population ratio. Hence, redefine that $x_i$ is population ratio of agents who choose to produce the $i^{th}$ kind of the goods.

As well as the price, the cost mechanism is defined as follows.

\[ c = c(x) = c_0 + Cx = (C_0 + C)x, \quad (c(x))_i \geq 0 \forall i, \]  
\[ c_0 \equiv c_0(1,1,\cdots,1)^T \in \mathbb{R}^m, \]  
\[ C_0 \equiv (c_0, c_0, \cdots, c_0) \in M_{m \times m}, \quad C \equiv (C_{ij}) \in M_{m \times m}. \]

Here, determine the conditions for economic plausibility on price and cost vectors. The following three conditions are required for economic plausibility on price, cost and profit vectors.

(a) For any production vector $x$, price, cost and profit must be always positive.
(b) All the orthogonal elements of price matrix $P$ should be negative.
(c) Non-orthogonal elements of price matrix $P$ and cost matrix $C$ should be at least sign symmetric.

\[ (a) \quad p_i(x) \geq c_i(x) \geq 0 \forall i \forall x, \]  
\[ (b) \quad P_{ii} \leq 0 \forall i, \]  
\[ (c) \quad \text{sgn}(P_{ij}) = \text{sgn}(P_{ji}), \quad \text{sgn}(C_{ij}) = \text{sgn}(C_{ji}) \forall i \forall j. \]

\text{sgn}(\ ) is the sign function; $(\text{sgn}(z)=1(z>0), \text{sgn}(z)=0(z=0), \text{sgn}(z)=-1(z<0))$.

### 2.2 Collective Learning Mechanism

It is natural that each agent is able to have neither a perfect knowledge on price structure matrix nor an optimal reaction for all the others. But they may be able to know current price, cost, and average profits as macro information of the market. So they are allowed to observe actual prices and change the kind of their production into the advantageous one. Hence, it is necessary to describe "micro-macro interaction" between individual decision of such "bounded rational" agents and macro status of market determined by amount of their decision.

An applicable modeling process of social attitude with micro-macro linkage is found as SLD (Social Learning Dynamics, Deguchi [3]). SLD is based upon not biological concepts of "random matching and selection" but Markov decision process. In SLD, state valuable $x_i$ means the probability of each micro agent chooses the social alternative "$i$", and it can be interpreted as the population ratio of agents that choose
the \( i \)th alternative. Hence, the state vector of SLD consists with that of extended Cournot-type price and cost mechanisms.

SLD considers that agents learn the landscape of social share \( x \) and expected profit \( E_i(x) \) for the \( i \)th social alternative (variation to produce). Here, \( W(x) \) represents the average of \( E_i(x) \). (4) The Markov decision process is defined by “\( q_{ij} \)”, which means the transition probability that an agent changes its choice from the \( i \)th to the \( j \)th alternative. (Fig. 1) Equation (5) is the fundamental assumption that the transition probability \( q_{ij} \) is proportional to both \( x_j \) (: how is the \( j \)th alternative supported by others?) and \( E_j(x)/W(x) \) (: how is the \( j \)th alternative profitable than the average?). Here the constants “\( r \)” indicates the given learning rate.

\[
E_i(x) \equiv p_i(x) - c_i(x) = \{(P_o - C_o + P - C)x_i\},
W(x) \equiv \sum_{i=1}^{n} E_i(x)x_i. \tag{4}
\]

\[
q_{ij} \equiv \Pr(i \rightarrow j|x) = r x_j E_j(x)/W(x). \tag{5}
\]

By the arguments in 2.1, \( E_i(x) \), \( W(x) \) and \( q_{ij} \) (for all \( i,j,x \)) can be derived from extended Cournot-type price and cost function. Now, SLD can be integrated with the market price-cost model.

Fig. 1. In the Markov decision process, \( q_{ij} \) means a transition probability that an agent changes its product from \( i \)th into \( j \)th kind of goods.

Taking continuous limit of Markov process in fig.1, the market dynamics is derived as following SLD (6).

\[
x_i(t + \Delta t) = x_i(t) + \left\{ \sum_j q_{ji}(x)x_j(t) - \sum_j q_{ij}(x)x_i(t) \right\} \Delta t
= x_i(t) + r \left\{ \sum_j E_j(x)x_j - \sum_j E_j(x)x_i \right\} \Delta t/W(x)
= x_i(t) + rx_i \Delta t \left\{ E_i(x) - W(x) \right\} / W(x).
\]

\[
dx_i/dt = r \frac{x_i \left\{ E_i(x) - W(x) \right\}}{W(x)} = rx_i \frac{(Rx)_i - x \cdot Rx}{x \cdot Rx}, \tag{6}
\]

\[
R \equiv P_o - C_o + P - C, \quad x \cdot y = x^T y.
\]
Here, the denominator of equation (6) does not depend on suffix “i”, so that an appropriate time scale can transform (6) into Replicator Dynamics by eliminating the denominator. (Deguchi[3]) Consequently, it realizes grounding of Social Learning Dynamics on particular market model. With little changes of market parameters, this market dynamics shows structural bifurcations, which mean the sudden disappearance of stable diversity or appearance of endogenous fluctuation. With focusing on such structural bifurcation, Deguchi presents indirect control and design of social institutions. (Deguchi [3]) As following section, we leave from Deguchi’s approach and try to explore about endogenous chaotic behavior and control methodology.

### 3 Chaotic Market Dynamics

This section illustrates that the market dynamics built at the former section shows chaotic bifurcations, which mean a slight change of particular parameters causes endogenous unpredictable fluctuation.

Any Replicator Dynamics needs at least 4 dimensions for chaos, since it conserves total population ratio (=1). In the evolitional game theory, Skyrms [5] investigated chaos on 4 by 4 game matrix. Here, we can demonstrate that our new market model shows bifurcation to chaos under an economically reasonable condition (3).

Assuming semi-independent two market segments (\(i=1\sim m\) and \(i=m+1\sim 2m\)) and set a bifurcation parameter reflecting integration or separation between these segments, the market dynamics can illustrate an aspect that market integration with changing price structure makes the market chaotic.

\[
\frac{dx_i}{dt} = r x_i \frac{(R x_i) - (x_i R x_i)}{(x_i R x_i)} (i = 1, 2, \cdots, 2m), \quad \sum_{i=1}^{2m} x_i (0) = 1,
\]

\[
R = P_0 - C_0 + P - C = P_0 - C_0 + (P^f + \gamma P^r) - (C^f + \gamma C^r),
\]

\[
P^f = \begin{pmatrix} p_1^f & 0 \\ 0 & p_2^f \end{pmatrix}, \quad P^r = \begin{pmatrix} 0 & p_1^r \\ p_2^r & 0 \end{pmatrix}, \quad C^f = \begin{pmatrix} c_1^f & 0 \\ 0 & c_2^f \end{pmatrix}, \quad C^r = \begin{pmatrix} 0 & c_1^r \\ c_2^r & 0 \end{pmatrix}.
\]

Equation (7) expresses that the price-cost matrices (: \(P, C\)) are composed of inner-segment price-cost structures (: \(P^f, C^f\)) and the inter-segment price-cost interactions (: \(P^r, C^r\)). The parameter \(\gamma\) connects these price-cost terms. For example, \(\gamma=0\) means that each segment is independent and large \(\gamma\) means that segments are more integrated and closely interacting.
Fig. 2. It's assumed that the market consists of two interacting inner segments. (e.g. $2m=4$)

\[
P^t = \begin{pmatrix} -0.75 & -0.9 & 0 & 0 \\ -0.1 & -0.75 & 0 & 0 \\ 0 & 0 & -0.5 & -0.2 \\ 0 & 0 & -0.1 & -0.75 \end{pmatrix}, \quad C^t = \begin{pmatrix} -0.3 & -0.1 & 0 & 0 \\ -1.2 & -0.5 & 0 & 0 \\ 0 & 0 & 0.15 & -1 \\ 0 & 0 & -0.2 & 0.3 \end{pmatrix}
\]

\[
P^e = \begin{pmatrix} 0 & 0 & -0.5 & 1 \\ 0 & 0 & -0.7 & -0.8 \\ -1 & -0.5 & 0 & 0 \\ 0.4 & -0.1 & 0 & 0 \end{pmatrix}, \quad C^e = \begin{pmatrix} 0 & 0 & 0.1 & 0.4 \\ 0 & 0 & -0.1 & 0.9 \\ 0.4 & -0.2 & 0 & 0 \\ 0.3 & 0.2 & 0 & 0 \end{pmatrix}
\]

\[
R = P_e - C_e + \left( P^t + \gamma P^e \right) - \left( C^t + \gamma C^e \right)
\]

\[
\begin{pmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{pmatrix} + \begin{pmatrix} -0.45 & -0.8 & 0 & 0 \\ 1.1 & -0.25 & 0 & 0 \\ 0 & 0 & -0.65 & 0.8 \\ 0 & 0 & -0.3 & -1.05 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & -0.6 & 0.6 \\ 0 & 0 & -0.6 & -1.7 \\ -1.4 & -0.3 & 0 & 0 \\ 0.1 & -0.3 & 0 & 0 \end{pmatrix}
\]

A numerical example (8) for (7) ($2m=4$) satisfies the economic conditions of (3). This market dynamics (7)-(8) illustrates bifurcation to chaos as follows.

Figure 3 shows the typical strange attractors after sufficient relaxation time.

Fig. 3. Profiles of a strange attractor of the market dynamics (6)-(7).

Figure 4 shows the bifurcation diagram has typical self-similarity in relatively wide range of interaction parameter gamma. Accordingly corresponding Liapunov exponent takes positive value.
This market model (7) is simple continuous dynamics that consists of the linear inverse demand function, linear cost function and Markov decision process without time-delay or folding-pie process. But, it shows chaotic behavior in the situation of the market increasing variation of goods and their interdependence of goods.

4. Control and Harnessing

For control and stabilization of chaotic agent system, we have presented KISS (Axelrod [6]) control agent like "sheepdog". (Kunigami & Trano [7]) This section briefly explains that such type of chaos control technique is also effective for harnessing the unpredictably complex behavior of the market.

As sheepdog controller, self-adapting delay-time feedback control (Kittel & Parisi & Pyragas [8]) is favorable for this research, since it is a continuous and simple
control method, which does not require the prior knowledge on the system. However, this method tends to choose quite unexpected orbit and to expand small interval fluctuation as delay time, so that the system sometimes becomes more unstable. Therefore, this research improves predictability of controller's behavior by setting control windows on both delay-time and macro objective function (e.g. average profit of market: W) value of controller. (Fig.6, Fig.7)

\[
\left[ \begin{array}{c} y(t) \\ y(t-\tau) \end{array} \right] = \begin{cases} f_i \cdot \frac{y_i(t-\tau) - y_i(t)}{|y_i(t-\tau) - y_i(t)|} & |y_i(t-\tau) - y_i(t)| > d_i \\ k_y \cdot (y_i(t-\tau) - y_i(t)) & |y_i(t-\tau) - y_i(t)| \leq d_i \end{cases}
\]

Fig. 6. The outline of delayed feedback control of chaotic system.

\[ \tau = \Delta t_{\text{peak}}^{(n)} - t_{\text{peak}}^{(n-1)} \]

Fig. 7. The window determines the thresholds of the control parameters.

. A numeric example illustrated in fig.8. With this control window, the self-adapting delay-time feedback control mechanism can stabilize the chaotic market behavior into periodic motion. On this stabilized periodic orbit, peak value and peak interval of the objective function are restricted beforehand by the windows.
Complexity and Control of Collective Learning Dynamics in a Simple Model of Market

**Fig. 8.** The self-adapting delay-time feedback control with the control window \((W_{TH}, \tau_{max}, \tau_{min})\) reduces complexity of the market dynamics \((7)(8)\).

On this control mechanism, several controllers can act simultaneously in the same market. Fig.9 illustrates that heterogeneous control agents can harness the chaotic market behavior. This example is suggestive so that two heterogeneous control agents with smaller control gains can more effectively reduce complexity of the market.

**Fig. 9.** Two heterogeneous control agents can harness the chaotic market \((7)(8)\).

5. Conclusions

The conclusions of this report are summarized as follows.

- We present simple but plausible market model with micro-macro interaction based on the integrating "Social Learning Dynamics" and Cournot type "inverse demand function". (Section 2)
• In spite of deriving from the linear inverse demand function, linear cost function and continuous time modeling, this market dynamics shows a bifurcation with proper chaotic behavior under an economically plausible condition. (Section 3)
• As “sheep dog” like agent, a chaos control technique can be applied to harness and stabilize the chaotic market dynamics. (Section 4)

Acknowledgements

We would like to specially thank Prof. Hiroshi Deguchi of Tokyo Institute of Technology, Prof. Tamotsu Onozaki of Aomori Public College, Dr. Hideki Takayasu of Sony Computer Science Laboratories for helpful insights and valuable discussions.

Reference