Recursive stability of social networks

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Very preliminary and incomplete.

Abstract

I propose a recursive generalization of pairwise stability of social and economic networks. My notion applies to many kinds of networks, encompassing the cooperative and the non-cooperative approach, allowing for multiple dimensions of interaction and for links with different intensity. Moreover, it explores recursively how different degrees of rationality are required for certain structures to form. Preliminary results are presented, which show that this notion is indeed manageable and may provide interesting insights.

1 Introduction

Economic interaction does not take place 'in vacuo'; rather, it is embedded in a social structure, as pointed out, among others, by Granovetter (1985). Recent research shows that in many contexts economic outcomes depend crucially on the details of this social structure and represent it as a network [see e.g. Kranton and Minehart (2001), Garicano (2000), Goyal and Moraga-González (2001), Goyal and Joshi (2001), Hendricks, Piccione and Tan (1995 and 1999), Cabrales and Calvò-Armengol (2001) and Eshel, Samuelson and Shaked (1998)]. A key feature of the general equilibrium approach is that it allows us to understand market dynamics without getting lost in the subtleties of such details. Though, exactly because of this reason, general equilibrium theory cannot explain some relevant interaction patterns. Game theory, on the other side, provides deep insights into the strategic dimension of economic interaction, but it generally lets individuals interact in an abstract setting and match randomly with one another. Indeed, real life matching processes are typically not random, and the amount and quality of information available, as well as the degree and the kind of rationality used in concrete situations, depend to a relevant extent upon the social context in which interaction takes place [see Myerson (1977)].

Sociologists have developed social network analysis as a mathematical tool to capture in a rigorous way some features of social structures [see Mitchell (1969) and Wassermann and Faust (1994)]. In the last decade economists have started to explore social networks as well, but from a different point of view, i.e. letting each node of a network be an actor, and studying games of network formation and games in networks[1] [see Jackson and Wolinsky (1996), henceforth JW96, and Bala and Goyal (2000), henceforth BG00, as seminal papers respectively for the cooperative and for the non-cooperative approach to network formation[2], Goyal and Vega-Redondo (1999) and Jackson and Watts (2000) for games of coordination and network formation together, and Chwe (2000) for a game of communication and coordination in social networks]. Although growing very fast, this literature is still at the beginning in comparison both with general equilibrium and with traditional game theory, but it seems a very promising route of research, the more so in a moment

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[1] This is also the difference between social and economic network theory on one side, and the operation research literature on how to optimize a network from a 'central planner' or an individual firm's point of view, on the other side.

[2] These two papers will serve us as a guide to some of the basic features in the literature.

in which even macroeconomists recognize the relevance of social structures for productivity, growth and welfare, as is shown by the recent literature on social capital.

One of the problems that arise within this approach is that networks become soon computationally difficult to manage. This is a problem both for theorists, although information technology renders it less and less relevant, and, more seriously, for individual actors in a network. In a very complicated environment, even if a full optimization were possible, its cost might exceed its benefits. So issues of bounded rationality become relevant. This has led scholars to study the stability of networks assuming boundedly rational agents. In particular, after JW96, the notion of pairwise stability (stability against deviations by a pair of agents) has become of common use in the literature on network formation and have given rise to a great number of related studies [see Jackson and Watts (1999), Jackson (2001), Bogomolnaia and Jackson (2001), Dutta and Jackson (2001), Haeringer (2000), Watts (2000), Jackson and van den Nouweland (2001), Bala and Goyal (2001)]. This notion assumes a low degree of rationality and provides a weak requirement for stability. Moreover, it applies only to a certain kind of networks.

Indeed, agent might find more complex ways to cooperative deviations than pairwise agreements, and networks may be different from the ones usually assumed. Starting with these ideas, I propose a recursive generalization of pairwise stability, with the hope of shedding light on the link between the degree of rationality, social structures and efficiency. My notion applies to many kinds of networks, encompassing the cooperative and the non-cooperative approach, allowing for multiple dimensions of interaction and for links with different intensity. Moreover, it explores recursively how different degrees of rationality are required for certain structures to form. At the present stage, I have just begun to study its possible applications. Preliminary results are presented, which show that this notion is indeed manageable and may provide interesting insights.

2 Economic interaction and social structure: the basic framework

What can be expected from the theory of economic networks is a framework that allows a good description of at least three main blocks: (i) the social structure in which individual interaction is embedded, (ii) the process of production and allocation of value, and (iii) their dynamic interplay. Until now, only a few special features of these three aspects have been explored in the literature. Progresses in this field can come basically from two ways: on a more 'applied' side, one can let a concrete situation drive the theoretical work; on a more 'abstract' side, one can try to improve the theoretical tools used in network analysis. These two ways are complementary and must ultimately proceed together, but in the present study only the second route is taken. I start spelling out a rather general formal framework.

2.1 Social structure

Let $N = \{1, ..., n\}$ be the (finite) set of players, $\Theta = \{1, ..., \vartheta\}$ be the (finite) set of player types, $\iota : N \to \Theta$ be a function that assigns a type to each player[3]. We suppose that players can interact with one another along many dimensions and with different intensities, and we let $R = \{1, ..., r\}$ be the (finite) set of relational dimensions and $M = \{0, ..., m\}$ be the (finite) set of possible relational intensities (one can easily imagine extensions of M). We propose three main interpretations of the strength of a link: it may indicate (i) frequency of interaction, (ii) volume of trade or (iii) intensity of affection (e.g. of trust or of friendship).

Let $N^A = \{(i, j) \in N \times N : i \neq j\}$ be the set of all ordered pairs of different players. A network is a function $g : N^A \to M^r : (i, j) \mapsto g_{ij} = (g^1_{ij}, ..., g^r_{ij})$ that specifies for each ordered pair of players a vector of relational intensities, i.e. the intensity of their link along all of their dimensions of interaction. Some of these dimensions may be of a symmetric kind, in the sense that the relational intensity is necessarily the same for both players (think e.g. of the frequency of direct communication with one another), whereas some other relations may be of an asymmetric kind, in the sense that the link of i with j ma y be different from the one of j with i. Thus, R can be partitioned into R^S and R^A, the sets of symmetric (non-directed) and asymmetric (directed) relations, respectively. Another possible way of thinking of R^S and R^A is in terms of two-way and one-way 'flow' 'communication' channels. Taking this partition into account, the set of all networks may be specified as $G = \{g : N^A \to M^r \mid \forall k \in R^S,\ \forall i, j \in$

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[3] A further generalization, not allowed for here, would be to consider individual affiliations to different groups.

$N : i \neq j, g^k_{ij} = g^k_{ji}\}$

Another partition of R concerns the possibility of strengthening and of weakening a link unilaterally vs. the need to find an agreement. Table 1 shows the possible combinations of unilateral and bilateral strengthening/weakening and defines the corresponding partition of R into R^{11}, R^{12}, R^{21} and R^{22}.

		weakening	
		one sided	two sided
strengthening	one sided	R^{11}	R^{12}
	two sided	R^{21}	R^{22}

Table 1

Some examples may help to get a feeling for these distinctions. For R^{11} (free choice of 'entry' and 'exit'), think e.g. of the frequency of connection to an internet site (or of writing e-mails or mails), or think of the intensity of trust or of friendship. It seems quite natural to think that $R^{11} \subseteq R^A$. For R^{21} (free 'exit', cooperative link formation), think e.g. of the frequency of direct communication or of direct meeting. R^{12} (unilateral strengthening, cooperative weakening) seems more a logical possibility that a relevant case. Finally, R^{22} corresponds to the case in which each player has veto power on strengthening or weakening a link: every change must be agreed upon.

This setup encompasses those of JW96 and of BG00 as special cases: JW96 amounts to assuming $r = \vartheta = 1,\ R = R^S = R21,\ M = \{0, 1\}$; BG00 assumes as well $r = \vartheta = 1$ and $M = \{0, 1\}$, but allows for the possibility of both $R = R^S = R21$ and $R = R^A = R11$.

2.2 Production and allocation of value

There are three main frameworks to think of value in a network context: (a) one can let the social structure be exogenously given and let value be endogenously produced and allocated by agents on the network; (b) conversely, one can study how the social structure changes endogenously for given patterns of interaction (or for a given value function); (c) finally, social structure and value may be seen as co-evolving[4].

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[4] One could tentatively speak of the social structure of value and of value of social structures.

JW96 adopts framework (b) and assumes that the value function and the allocation rule are given. Since the authors do not specify them, this approach is very general. BG00 consider a subcase in which an individual's payoff increases with the number of players accessible to him or her and decreases with the number of links directly supported by him or her.

2.3 Time and dynamics

The timing of decision making and the dynamics of network changes are probably the most problematic aspects to capture in a general framework, because their specification is highly dependent upon the specific situation one is interested to model. Nevertheless, the literature on this topic has so far adopted a quite restricted number of solutions, in part for their analytical tractability.

JW96 focuses on the main aspect R^{21} type relations, i.e. that "the formation of a link requires the consent of both parties involved, but severance can be done unilaterally" and consequently investigates pairwise stability. The full step towards a dynamic model is made in Jackson and Watts (1999), where myopic improving paths from one network to another are studied, and then their stochastic stability is analyzed, on a way that leads to evolutionary cooperative game theory. BG00 uses as well the notion of stochastic stability, but starting from the Nash equilibrium of the stage 'network game'.

3 Recursive stability

Let the sets N (players),Θ (player types),R (interaction dimensions), M (possible relational intensities), N^A (oriented pairs of actors) and G (possible networks) be defined as above, and let the value function $v : G \to \Re$ be given (V being the set of all possible value functions), as well as the allocation rule $Y : G \times V \to \Re^n$. To begin with, let us make three simplifying assumptions.

Assumption 1 $\theta = 1$. *All players are equal.*

Assumption 2 $R^A = R^{11}$ *(hence,* $R^S = R^{12} \bigcup R^{21} \bigcup R^{22}$*). Some degree of bilateral cooperation is needed in all symmetric relations. Conversely, non symmetric relations can be unilaterally changed.*

Assumption 3 $R^{12} = \emptyset$. *No relations can be strengthened unilaterally, but weakened only with a bilateral consensus.*

Assumptions 1 and 3, at the present stage, are only useful to simplify notation. Assumption 2, on the contrary, is relevant, but seems also quite natural.

Let us now construct a recursive definition of stability, that allows for a variety of deviation possibilities.

For $g \in G$, $i \in N$, $\xi_i \subseteq N$, let $C_i(\xi_i) \equiv \{(i,j) \in N^A \ : \ j \in \xi_i\}$ be the set of i's links with the players in ξ_i, and let $G_i(g,\xi) \equiv \{\tilde{g} \in G \ : \ \tilde{g}|_{N^A \backslash C_i(\xi_i)} = g|_{N^A \backslash C_i(\xi_i)} \ \wedge \ \forall j \in \xi_i \ : \ j \neq i, \ [\forall p \in R^{21} \ \tilde{g}_{ij}^p \leq g_{ij}^p, \ \forall q \in R^{22} \ \tilde{g}_{ij}^q = g_{ij}^q]\}$ be the set of networks achievable by player i starting from g through a unilateral change of his or her links with the players in ξ_i, given that no other player revises his or her links at the same time. With this notation, it is easy to give the following two definitions.

Definition 1 *A network $g \in G$ is individually one-link stable (which we indicate with (1,1)-S, or simply 1-S) with respect to v and Y if $\forall i,j \in N \ : \ i \neq j$, $\forall g' \in G_i(g,\{j\})$, $Y_i(g',v) \leq Y_i(g,v)$. In words, a network is individually one-link stable if no individual has the possibility of being strictly better off through a unilateral change of a single link, given that all other links remain unchanged.*

Definition 2 *A network $g \in G$ is individually k-link stable (which we indicate with (1,k)-S) with respect to v and Y if $\forall i \in N$, $\forall \xi_i \subseteq N \backslash \{i\} \ : \ \sharp(\xi_i) = k$, $\forall g' \in G_i(g,\xi_i)$, $Y_i(g',v) \leq Y_i(g,v)$. In words, a network is individually k-link stable if no individual has the possibility of being strictly better off through a unilateral change of k of his or her links, given that all other links remain unchanged.*

The same idea extends to deviations by coalitions. Given a network $g \in G$ and a coalition $\nu \subseteq N$, let $C(\nu) \equiv \cup_{j \in \nu} C_j(\nu)$ be the set of intra-coalition links, and let $G_\nu(g,\nu) \equiv \{\tilde{g} \in G \ : \ \tilde{g}|_{N^A \backslash C(\nu)} = g|_{N^A \backslash C(\nu)} \ \wedge \ \forall i \in \nu, \tilde{g} \notin G_i(g,\nu)\}$ be the set of networks achievable by coalition ν starting from g through a cooperative decision on $C(\nu)$, given that all other links remain unchanged. We can now extend individual stability to coalition stability.

Definition 3 *A network $g \in G$ is λ-wise stable with respect to v and Y (which we indicate with (λ,0)-S, or simply λ-S, where $\lambda \in \{2,...,n\}$) if the two following conditions are satisfied:*
(i) g is (λ-1,1)-S (defined below);

(ii) $\forall \nu \subseteq N \,:\, \sharp(\nu) = \lambda,\ \forall g' \in G_\nu(g,\nu),\ [\exists i \in \nu \,:\, Y_i(g',v) > Y_i(g,v) \Rightarrow \exists j \in \nu \,:\, Y_j(g',v) < Y_j(g,v)]$.
In words, a network is λ-wise stable if it is (λ-1,1)-S and if no coalition of λ players is able to reach a Pareto-improvement for its members through cooperation (i.e., through a cooperative decision on how to simultaneously change the intra-coalition links), given that all extra-coalition links remain unchanged.

In the above definition we have left the predicate (λ-1,1)-S undefined. Let us now complete our recursive definition of network stability by allowing for simultaneous deviations both cooperatively by coalitions and unilaterally by individual players. We need a last piece of notation. Given a network $g \in G$ and a coalition $\nu \subseteq N$, let $\xi(\nu) \equiv \xi_i \in N\backslash\nu \;:\; i \in \nu$ be a collection of coalition ν members' extra-coalition 'partner sets' ($j \in \xi_i$ means that player i may think of unilaterally changing the link g_{ij}). For each coalition member $i \in \nu$, let moreover $C_i = C_i(\nu, \xi_i) \equiv \{(i,j) \in N^A \;:\; j \in \nu \cup \xi_i\}$ be the set of i's potentially changeable links within the coalition and within his or her extra-coalition 'partner set' ξ_i. Let $C = C(\nu, \xi(\nu)) \equiv \cup_{\iota \in \nu} C_\iota = \{(\iota, j) \in N^A \;:\; \iota \in \nu,\ j \in \nu \cup \xi_\iota\}$ be the set of links changeable through a coalition's agreement. Call $G_i = G_i(g,\nu,\xi_i) \equiv \{\tilde{g} \in G \;:\; \tilde{g}|_{N^A \backslash C_i} = g|_{N^A \backslash C_i} \;\wedge\; \forall j \in \nu \cup \xi_i \,:\, j \neq i,\ [\forall p \in R^{21}\ \tilde{g}^p_{ij} \leq g^p_{ij},\ \forall q \in R^{22}\ \tilde{g}^q_{ij} = g^q_{ij}]\}$ the set of unilaterally feasible networks for player i. Let $G_\nu = G_\nu(g,\nu,\xi(\nu)) \equiv \{\tilde{g} \in G \;:\; \tilde{g}|_{N^A \backslash C} = g|_{N^A \backslash C} \;\wedge\; \forall i \in \nu \tilde{g} \notin G_i\}$ be the set of cooperatively feasible networks for coalition ν (notice that this is a narrow definition, since this set is defined such that cooperation is necessary and not just sufficient). We can now give the following definition.

Definition 4 *A network $g \in G$ is λ-wise k-link stable with respect to v and Y (in short, (λ,k)-S, where $\lambda \in \{2,...,n\}$ and $k \in \{0,...,n-\lambda\}$) if the two following conditions are satisfied:*
(i) g is (λ-1,k+1)-S;
(ii) $\forall \nu \subseteq N \,:\, \sharp(\nu) = \lambda,\ \forall \xi(\nu) \,:\, \sharp(\xi_1) = ... = \sharp(\xi_\lambda) = k,\ \forall g' \in G_\nu,\ [\exists i \in \nu \,:\, Y_i(g',v) > Y_i(g,v) \Rightarrow \exists j \in \nu \,:\, Y_j(g',v) < Y_j(g,v)]$.
In words, a network is λ-wise k-link stable if it is (λ-1,k+1)-S and if no coalition of λ players is able to reach a Pareto-improvement for its members through cooperation, even if each of them is simultaneously allowed to unilaterally change k of his or her links with players that do not belong to the coalition.

Notice that the computational difficulty for players increases very fast with k. Moreover, as λ increases, it becomes more and more difficult, and possibly more and more costly, to reach an agreement among all the members of a coalition of λ players. To get a feeling of what our recursive definition means, consider pairwise k-link stability. As k increases, already this notion becomes very restrictive, since it excludes, first of all, any possibility of unilateral improvement, when a single player can try to change unilaterally $k+1$ of his or her links (i.e., (2,k)-S ⇒ (1,k+1)-S); secondly, it excludes any possibility of Pareto-improvement for a couple, even when both players are allowed to simultaneously change the link between them and and other k links each, and given that they can reach an agreement between themselves on all these changes, but they cannot establish multilateral agreements. The idea here is to treat a simultaneous change in the links of the two players as a form of agreement.

Figure 1 helps to understand the recursive construction of our stability concept.

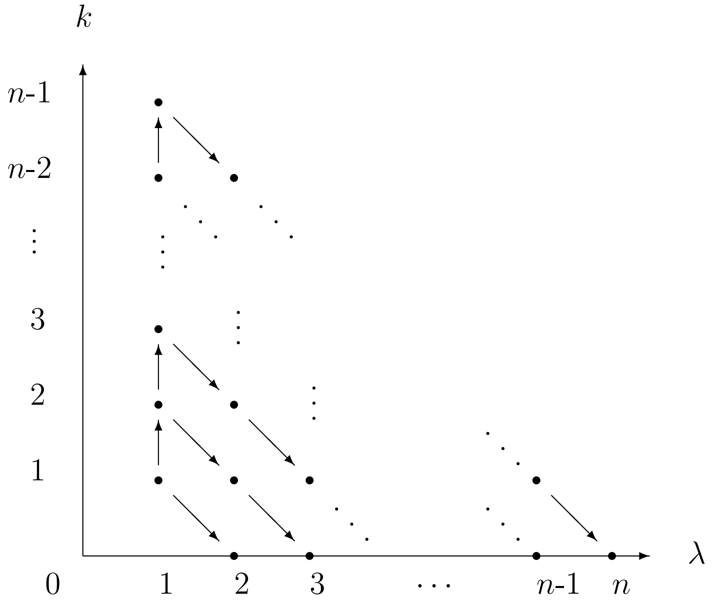

Figure 1

Notice that $a \rightarrow b$ indicates here the procedure of recursive construction, and means that a is necessary for b. So the logical implication is $b \Rightarrow a$. We

can establish the following properties, that follow from the definition: for $\lambda \in \{1, ..., n\}$ and $k \in \{0, ..., n-\lambda\}$,
(i) (λ,k)-S $\Rightarrow$ (λ,h)-S $\quad \forall h \in \{0, ..., k\}$;
(ii) (λ,k)-S $\Rightarrow$ (h,k)-S $\quad \forall h \in \{1, ..., \lambda\}$;
(iii) (λ,k)-S $\Rightarrow$ (λ-h,k+h)-S $\quad \forall h \in \{0, ..., \lambda - 1\}$.

There is a strict connection between our recursive definition of stability and the debate in the literature about the choice between a cooperative and a noncooperative approach and about the efficiency properties of stable networks. This connection is established as follows.

Proposition 1 *Without costs of establishing links and of forming coalitions, and with full rationality, a network $g \in G$ is (1,n-1)-S if and only if it corresponds to a Nash equilibrium of a noncooperative game of network formation.*

Proposition 2 *A network $g \in G$ is (n,0)-S if and only if it is efficient.*

Notice that in the latter proposition the concept of efficiency does not necessarily mean strong efficiency: g is not necessarily a maximizer of the value function v, but it is at least a maximizer of the aggregate allocation $\sum_{i \in N} Y_i(g, v)$. Under some assumptions on the allocation rule, it should be possible to obtain that g is (n,0)-S if and only if it is strongly efficient.

The framework outlined raises a number of questions. The first one concerns its practical applicability; second, since it encompasses and generalizes different approaches, one might ask which of the results obtained so far in the literature may be generalized and which other ones cannot. A preliminary exploration indicates, for instance, that Lemma 1 in Jackson and Watts (1999) holds true for every λ and k. This lemma says that for any value function v and allocation function Y there exists at least one pairwise stable network or closed cycle of networks. Indeed, there there exists at least one (λ, k)-S network or (λ, k)-closed cycle of networks. We do not go more in depth here in stating correctly and proving this assertion.

The degree to which some results hold true might depend to a crucial extent upon the hypotheses one is willing to make about the degree of players' rationality and about the costs associated to changing links and forming coalitions. A further step would be to look at the dynamics of networks where agents have 'realistic' action possibilities. When we know something more about these aspects, it will be possible to remove the key limitation of the approach followed here, i.e. the fact that we are only looking at network formation *per se*, without taking into account its interplay with other forms of economic interaction, as discussed in the previous section. Let us start with the first step of such research program: a ‘concrete’ application.

4 Application: the connection model

The basic connection model is presented by Jackson and Wolinsky (1996). Agents attribute a positive value to other agents’information, but, in order to be able to acquire it, they need to be linked to one another, either directly or through some intermediate agents. The longer the chain of agents one has to go through to reach some information, the worse the quality of the information received. The presence of a positive cost of forming direct connections implies that the best for an agent would be to have only one link with a central agent, who, in turn, is directly connected with everybody else. In this case, the network would assume the form of a star. The problem is that, of course, nobody has an incentive to become a central agent. In the basic model, there is only one type of agent ($\vartheta = 1$), only one dimension of interaction ($r = 1$), namely communication, which is symmetric ($R = R^S$), so that we have non-directed graphs, and such that agreement is necessary to open a communication channel, but a unilateral decision is sufficient to close it ($R = R^{21}$). Finally, agents are either linked to one another or not linked at all ($M = \{0, 1\}$). We extend the model by allowing different degrees (or frequencies, or intensities) of communication: we let $M = \{0, \frac{1}{m}, ..., \frac{m-1}{m}, 1\}$.

Let b_{ij} be the intrinsic value for i of j’s information and c_{ij} be the cost for i of the link g_{ij}. We study the symmetric case in which $\forall i, j \in N : i \neq j,\ b_{ij} = 1$ and $c_{ij} = cg_{ij}$, where $c > 0$ is a constant ($b_{ii} = c_{ii} = 0$). Given two agents i and j, a path from i to j in a network g, p_{ij}, is a sequence of (different) agents $\{n_0, ..., n_H\}$ such that $n_0 = i,\ n_H = j$, and $\forall h \in \{0, ..., H-1\},\ g_{n_h n_{h+1}} > 0$. We call $P_{ij}(g)$ the set of all paths from i to j in g. The length of a path is the number of its links and is given by a length function $t : P_{ij}(g) \to \{1, ..., n-1\} : p_{ij} \mapsto t(p_{ij}) = \sharp(p_{ij}) - 1$. The amount of information that can flow through a path is conditioned by the weakest link, i.e. by the link with the lowest intensity of communication along the path. Thus, we define the value of a path as $w : P_{ij}(g) \to M \backslash \{0\} : p_{ij} \mapsto w(p_{ij}) = \min_{h \in \{0,...H-1\}} \{g_{n_h n_{h+1}} : n_h, n_{h+1} \in p_{ij}\}$. The length of the shortest path from i to j is called geodesic distance between i and j in g

and is denoted by $t_{ij}(g) = \min_{p_{ij} \in P_{ij}(g)} t(p_{ij})$. The value of j for i in g is the maximum amount of j's information that can flow to i through one of the available paths: $w_{ij}(g) = \max_{p_{ij} \in P_{ij}(g)} w(p_{ij})$. The combined effect of length and value gives us the discounted value of a path: $w^d(p_{ij}) = \delta^{t(p_{ij})} w(p_{ij})$, where $\delta \in (0, 1]$. The discounted value of j for i in g is finally

$$w^d_{ij}(g) = \begin{cases} \max_{p_{ij} \in P_{ij}(g)} w^d(p_{ij}), & \text{if} \quad \exists\, p_{ij} \in P_{ij}(g) \\ 0 \qquad\qquad\qquad\quad , & \text{if} \quad \nexists\, p_{ij} \in P_{ij}(g) \end{cases}$$

We are now able to write down players'utility: $u_i(g) = \sum_{j \neq i}[w^d_{ij}(g) - cg_{ij}]$. The value of a network will then be given by $v(g) = \sum_{i \in N} u_i(g)$.

A network g is said to be strongly efficient if $\forall g' \in G,\ v(g) \geq v(g')$; it is said to be Pareto-efficient if there does not exist a $g' \in G$ such that $\forall i,\ u_i(g') \geq u_i(g)$ and $\exists\, j\ :\ u_j(g') > u_j(g)$.

Let us study the relation between stability and efficiency.

... quote JW's results ...

Since we are interested in the process of network formation, it seems natural to start with the empty network (denoted by $\emptyset \in G$ such that $\forall (i, j) \in N^A,\ \emptyset_{ij} = 0$) and study the conditions under which individuals will be willing to establish links, as well as the kind of links that will be established.

Proposition 3

(i) *The empty network* $\emptyset$ *is (1,k)-S* $\quad \forall k \in \{1, ..., n-1\}$;

(ii) $\emptyset$ *is (2,k)-S* $\quad \forall k \in \{0, ..., n-2\} \quad \Leftrightarrow \quad \delta \leq c$;

(iii) $\emptyset$ *is (3,k)-S* $\quad \forall k \in \{0, ..., n-3\} \quad \Leftrightarrow \quad \delta < c$.

Proof

(i) This follows trivially form $R = R^{21}$ and the definition of $\emptyset$: no player can unilaterally weaken non-existing links.

(ii) Notice first that, by the same argument of point (i), in the case of $\emptyset$ unilateral deviations are irrelevant, so it is enough to prove that $\emptyset$ is (2,0)-S $\Leftrightarrow$ $\delta \leq c$. Now, $\emptyset$ is (2,0)-S if and only if it is not Pareto-dominated by any network g obtained from $\emptyset$ by adding a single link $g_{ij} = g_{ji} > 0$ (we write in short that there is no g that (2,0)-D $\emptyset$). Thus, it must not be the case that $u_i(g) > u_i(\emptyset)$ and $u_j(g) \geq U_j(\emptyset)$. Since $u_i(g) > u_i(\emptyset) = 0 \Leftrightarrow \sum_{k \neq i}[w^d_{ik}(g) - cg_{ik}] = \delta g_{ij} - cg_{ij} = (\delta - c)g_{ij} > 0 \Leftrightarrow \delta > c$, the result follows.

(iii) By point (i), it is enough to prove that $\emptyset$ is (3,0)-S. Now, $\emptyset$ is (3,0)-S if and only if it is (2,1)-S and it is not Pareto-dominated by any network g obtained from $\emptyset$ by adding links within a coalition ν of 3 players (i.e., there is no g that (3,0)-D $\emptyset$). By point (ii), we know that $\emptyset$ is (2,1)-S if only if $\delta \leq c$. So assume $\delta \leq c$.

We now prove that g (3,0)-D $\emptyset$ if and only if the two following conditions are satisfied:

- **(a)** $\delta = c \in (0, 1]$, and
- **(b)** g is such that $\exists\, \nu = \{i, j, k\} \subseteq N \ : \ g|_{N^A \setminus C(\nu)} = \emptyset|_{N^A \setminus C(\nu)}$ and $0 \leq g_{ik} < \delta g_{ij} \leq g_{ij} \leq g_{jk} \leq 1$.

Suppose (a) and (b) hold. Under these conditions, g (3,0)-D $\emptyset \Leftrightarrow \forall \iota \in \nu,\ u_\iota(g) \geq 0 \ \wedge \ \exists\, \iota \in \nu \ : \ u_\iota(g) > 0$. Let us check this.

$$\begin{aligned}
u_i(g) &= [\max\{\delta g_{ij},\, \delta^2 g_{ik}\} - c g_{ij}] + [\max\{\delta g_{ik},\, \delta^2 g_{ij}\} - c g_{ik}] = \\
&= (\delta - c) g_{ij} + \delta^2 g_{ij} - c g_{ik} = \delta(\delta g_{ij} - g_{ik}) > 0 \\
u_j(g) &= [\max\{\delta g_{ij},\, \delta^2 g_{ik}\} - c g_{ij}] + [\max\{\delta g_{jk},\, \delta^2 g_{ik}\} - c g_{jk}] = \\
&= (\delta - c)(g_{ij} + g_{jk}) = 0 \\
u_k(g) &= [\max\{\delta g_{ik},\, \delta^2 g_{ij}\} - c g_{ik}] + [\max\{\delta g_{jk},\, \delta^2 g_{ik}\} - c g_{jk}] = \\
&= \delta^2 g_{ij} - c g_{ik} + (\delta - c) g_{jk} = \delta(\delta g_{ij} - g_{ik}) > 0
\end{aligned}$$

Now suppose that (b) holds, but (a) does not. We show that this contradicts g (3,0)-D $\emptyset$. In fact, given $\delta \leq c$, not (a) means $\delta < c$. Then $u_i(g) < 0$.

This completes the proof, because we have shown that if $\delta \geq c$, then $\emptyset$ is not (3,0)-S (i.e., $\emptyset$ (3,0)-S $\Rightarrow$ $\delta < c$), and if $\delta < c$, then $\emptyset$ is (3,0)-S.

The proof of point (iii) is constructive and deserves some comments. It shows how a network has to be in order to Pareto-dominate the empty network when only deviations by coalitions of three players are allowed: it has to be highly asymmetric, in the sense that two of the new links must be much stronger than the third one. The player strongly connected with the other two ones will be indifferent (because it must hold $\delta = c$), whereas the other two ones will strictly gain, given that they have to pay lower costs. In particular, within the class of the networks g such that g (3,0)-D $\emptyset$, the network

$\tilde{g} : \tilde{g}_{ik} = 0, \ \tilde{g}_{ij} = \tilde{g}_{jk} = 1$ is Pareto-efficient, because it maximizes $u_i(g)$ and $u_k(g)$, keeping $u_j(g) = 0$ (notice that this is a triangle with a missing side).

What do we learn from the above proposition? In a way, very little, because the empty network is the least interesting one and we do not even fully characterize (λ,k)-stability, but for $\lambda = 1, 2, 3$. On the other side, this exercise shows that a characterization is indeed possible and thus opens the way to further research. Moreover, it shows constructively how asymmetry may arise in an endogenous process of network formation. This result, although elementary, allows some speculation. First of all, the constraint we impose through λ and k might be due to bounded rationality or to some 'connection technology', that imposes a cost on coalition formation and on unilateral changes. In any case, it becomes interesting to study how it affects the process of network formation. If we stop at the level of pairwise stability, we may miss some relevant issues. Moreover, asymmetry, besides bounded rationality (or 'connection technology') may add a bargaining cost for the best position, which renders difficult the agreement. Indeed, these are just some preliminary notes, better suited to illustrate how a model like this may be interesting and worth of further research than providing any relevant result themselves. As soon as λ and k increase, the number of possible combinations becomes very high, so that it becomes worthwhile to explore them letting a computer run some algorithm.

5 Conclusion

To be written.

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