Dynamic Oligopoly with Capital Accumulation and Environmental Externality*

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Abstract

We model the interplay between capital accumulation for production and environmental externalities in a differential oligopoly game with Ramsey dynamics. The external effect is determined, alternatively, by sales or production. While the externality does not affect the behaviour of profit-seeking firms, it may induce a benevolent planner to shrink sales as compared to the Cournot-Nash equilibrium because of a tradeoff between consumer surplus and the externality, if the latter is driven by sales. If instead it is determined by production, there emerges that the Ramsey golden rule is no longer socially optimal.

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1 Introduction

The control of polluting emissions damaging the environment is a hot issue and is receiving an increasing amount of attention in the current literature in the field of environmental economics. Most of the existing contributions investigate the design of optimal Pigouvian taxation aimed at inducing firms to reduce damaging emissions, both in monopoly and oligopoly settings.[1] The established approach to this problem consists in taking the social optimum, where a benevolent planner chooses a production plan for the firms in the industry so as to maximise social welfare, as a benchmark against which the performance of the profit-seeking firms has to be assessed. This produces corrective policy measures which, ideally, should take the form of tax schemes able to reproduce the same social welfare level associated with the first best.

Another stream of literature analyses the feasibility of tradeable pollution permits, which, however, may lead to the monopolization of the industry.[2]

To the best of our knowledge, the interplay between environmental externalities and capital accumulation under oligopoly or imperfect competition has received scanty, if any, attention thus far.[3] Indeed, this is a relevant facet of the general matter, in particular in view of the current debate on globalization and the ambiguous attitude adopted in this respect by new major actors, like China and India, but also Brazil, in shaping the current look of the international economic system for the new millennium.

We illustrate a dynamic oligopoly model where firms accumulate capacity *à la* Ramsey (as in Cellini and Lambertini, 1998, 2008) to produce the final good and either sales or production cause a negative environmental externality (pollution). Given the assumption that firms do not internalise the externality, the latter does not affect their optimal be-

[1] See Bergstrom *et al.* (1987), Karp and Livernois (1992, 1994) and Benchekroun and Long (1998, 2002), *inter alia*.

[2] To this regard, see Newbery (1990) and von der Fehr (1993), *inter alia*.

[3] Instead, uncountably many contributions studying the interplay between pollution and growth or technical change do exist. See Bovenberg and de Mooij (1997), Hartman and Kwon (2005), Jouvet *et al.* (2005), Dutta and Radner (2006), Greiner (2007), Ricci (2007), Bartz and Kelly (2008), Itaya (2008). for an overview, see Dockner *et al.* (2000).

haviour, yielding either the Cournot-Nash solution or the Ramsey golden rule as a saddle point equilibrium, depending upon the relative size of parameters. Clearly, the opposite holds at the social optimum, where the maximization of social welfare also accounts for the external effect. If the externality depends on sales, then a benevolent social planner may find it convenient to produce less than the profit-seeking firms if the weight attached to the externality is sufficiently high, in view of the tradeoff between the externality itself and consumer surplus. When instead the external effect is determined by production, the picture of the profit-seeking behaviour remains the same while the social optimum changes drastically, with the Ramsey golden rule disappearing as a stand-alone equilibrium.

The basic model is in Section 2. Section 3 contains the oligopoly game among profit-seeking firms, while the analysis of the social optimum in the case where the externality is determined by sales is in Section 4. The comparative analysis of the two regimes is carried out in Section 5. The alternative model where the externality depends on production is laid out in Section 6. Concluding remarks are in section 7.

2 The set up

The present set up is a simplified version of the dynamic game presented in Cellini and Lambertini (1998). Consider a market where N identical firms produce and sell a homogeneous product under Cournot competition. The inverse demand function for the good is

$$p(t) = a - \sum_{i=1}^{N} q_i(t) \tag{1}$$

where $q_i(t) \in [0, \overline{q})$ is the quantity produced and sold by firm i at time t and $a > 0$ is the reservation price. Production costs are linear in quantities

$$C_i(t) = cq_i(t)$$

with $c \geq 0$ being exogenously given and identical for all firms.

Production requires physical capital $k_i(t)$ that accumulates over time to create capacity. At any instant of time t, the output level of each firm is

$$y_i(t) = f(k_i(t)) \tag{2}$$

where $f' \equiv \partial f(k_i(t))/\partial k_i(t) > 0$ and $f'' \equiv \partial^2 f(k_i(t))/\partial k_i^2(t) < 0$. We assume that, at any time t, $q_i(t) \leq y_i(t)$, so that the level of sales cannot exceed the quantity produced. Output that is not sold is used to build up productive capacity according to

$$\dot{k}_i(t) = f(k_i(t)) - q_i(t) - \delta k_i(t) \tag{3}$$

where $\delta > 0$ is the depreciation rate of capital.

Under the above assumptions, the instantaneous profit of each firm is $\pi_i(t) = (p(t) - c)q_i(t)$. Given a common intertemporal discount rate $\rho > 0$, the goal of each firm is to maximize the discounted value of its flow of profits

$$\Pi_i = \int_0^\infty e^{-\rho t}\pi_i(t)dt \tag{4}$$

under the dynamic constraint (3).

With respect to Cellini and Lambertini (1998), we now introduce the assumption that producing the good is polluting and that this externality is not taken into account by the single firm (which is myopic or simply not interested in this aspect of its activities), but it enters the social welfare evaluation made by a benevolent social planner. Assuming that the social cost of pollution at any time t is quadratic in the total amount of output sold, the social welfare function of the social planner is

$$sw(t) = \sum_{i=1}^{N} \pi_i(t) + CS(t) - EXT(t) \tag{5}$$

$$EXT(t) = \beta \left[\sum_{i=1}^{N} q_i(t) \right]^2$$

where the first term represents the profits of the N firms, $CS(t) = (a - p(t))Q(t)/2 = Q(t)^2/2$ is consumer surplus and the last term represents the social cost of pollution, with

$\beta > 0$. Observe that here the environmental externality $EXT(t)$ depends on actual sales (or equivalently, consumption) and not on production or installed capacity. In the remainder we will also discuss the alternative cases where either $EXT(t) = \beta \left[\sum_{i=1}^{N} f(k_i(t))\right]^2$ or $EXT(t) = \beta \left[\sum_{i=1}^{N} k_i(t)\right]^2$.

In next section we determine the open-loop Nash equilibrium of the game played by N firms neglecting the social cost of pollution. Then we compare this equilibrium with the solution that would be implemented by a benevolent social planner that takes into consideration also the social cost of pollution.

3 Cournot competition

Given that the model is not built in linear-quadratic form, we will focus our attention on the open-loop solution. The current-value Hamiltonian function of each firm i is

$$H_i(t) = \pi_i(t) + \lambda_{ii}(t)\dot{k}_i(t) + \sum_{j \neq i} \lambda_{ij}(t)\dot{k}_j(t) \tag{6}$$

$$= \left[a - c - q_i(t) - \sum_{j \neq i} q_j(t)\right] q_i(t) + \lambda_{ii}(t)\left[f(k_i(t)) - q_i(t) - \delta k_i(t)\right] + \sum_{j \neq i} \lambda_{ij}(t)\left[f(k_j(t)) - q_j(t) - \delta k_j(t)\right] \tag{7}$$

where $\lambda_{ii}(t)$ is the costate variable associated to $k_i(t)$ and $\lambda_{ij}(t)$ is the costate variable associated to $k_j(t)$ by firm i. The initial condition for firm i is $k_i(0) = k_{i0}$.

Under the requirement that the following set of transversality conditions

$$\lim_{t \to \infty} \lambda_{ij}(t)k_j(t) = 0 \tag{8}$$

is satisfied for all i and j, the necessary conditions for a path to be optimal are:

$$\begin{aligned}
\frac{\partial H_i}{\partial q_i} &= a - c - 2q_i - \sum_{j \neq i} q_j - \lambda_{ii} = 0 && (9)\\
-\frac{\partial H_i}{\partial k_i} &= -\lambda_{ii}\,[f' - \delta] = \dot{\lambda}_{ii} - \rho\lambda_{ii} \Rightarrow \dot{\lambda}_{ii} = \lambda_{ii}[\rho + \delta - f'] && (10)\\
-\frac{\partial H_i}{\partial k_j} &= -\lambda_{ij}\left[\frac{\partial f(k_j)}{\partial k_j} - \delta\right] = \dot{\lambda}_{ij} - \rho\lambda_{ij}. && (11)
\end{aligned}$$

where the time arguments are omitted for brevity. From (9) we obtain, for all t,

$$q_i = \frac{1}{2}\left[a - c - \sum_{j \neq i} q_j - \lambda_{ii}\right]. \tag{12}$$

As the costate variables λ_{ij} are irrelevant for the optimal choice of sales q_i (indeed any costate eq. (11) is in separable variables and admits the solution $\lambda_{ij} = 0$ at all times), we proceed by setting $\lambda_{ij}(t) = 0$ for all $i \neq j$ and all t. Differentiating (12) with respect to time and using (9-10), we get

$$\dot{q}_i = -\frac{1}{2}\left[\sum_{j \neq i} \dot{q}_j + \dot{\lambda}_{ii}\right] = -\frac{1}{2}\left[\sum_{j \neq i} \dot{q}_j + (a - c - 2q_i - \sum_{j \neq i} q_j)(\delta + \rho - f')\right]. \tag{13}$$

Given the ex-ante symmetry, we impose that the choices made by the firms are symmetrical, i.e.:

$$q_i = q_j = q \quad \forall j \neq i, \forall t \tag{14}$$

Under the above assumption, (13) simplifies to

$$(N+1)\dot{q} = [a - c - (N+1)q]\,(f' - \delta - \rho). \tag{15}$$

The state-control dynamic system of the model is the following one:

$$\begin{cases} \dot{k} = f(k) - q - \delta k \\ \dot{q} = \dfrac{1}{N+1}\,[a - c - (N+1)q]\,(f'(k) - \delta - \rho) \end{cases} \tag{16}$$

The steady state pair (k, q) solves one of the following systems

$$\begin{cases} q^C = \dfrac{a-c}{N+1} \\ q^C = f(k^C) - \delta k^C \end{cases} \tag{17}$$

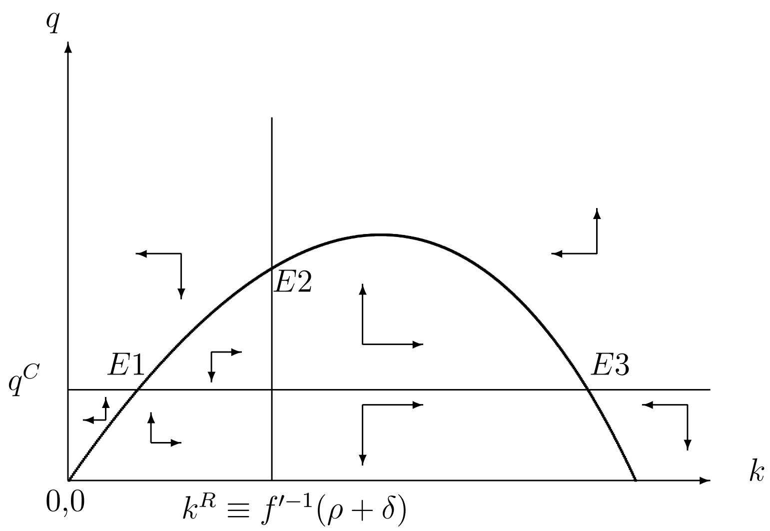
$$\begin{cases} f'(k^R) = \delta + \rho \\ q^R = f(k^R) - \delta k^R \end{cases} \tag{18}$$

where the first solution is the familiar Ramsey golden rule solution and the second one represents the static solution that emerges from the static version of the Cournot game. For further reference, the total output produced under the Cournot solution is

$$Q^C = Nq^C = \frac{N(a-c)}{N+1}. \tag{19}$$

To visualize one possible solution of the game, consider Figure 1.

Figure 1: The phase diagram under Cournot competition



The locus $\dot{k} \equiv dk/dt = 0$, as well as the dynamics of k (depicted by the horizontal arrows), derives from the first equation of system (16). The locus $\dot{q} \equiv dq/dt = 0$ is given by the solutions of the second equation of (16) and consists of a horizontal branch (corresponding to the Cournot solution q^C) and of a vertical branch (corresponding to the Ramsey solution where $f'(k^R) = \delta + \rho$). The dynamics of q is summarised by the vertical arrows. Steady state equilibria, denoted by $E1$, $E3$ along the horizontal branch, and $E2$ along the vertical one, are identified by the intersections between loci. Notice that, as $E1$ and $E3$ entail the same levels of sales, point $E3$ is inefficient in that it requires a higher amount of capital.

Figure 1 describes only one out of five possible configurations, due to the fact that the position of the vertical line $f'(k) = \rho + \delta$ is independent of demand parameters, while the locus $q^C = (a - c) / (N + 1)$ shifts upwards (resp., downwards) as a (resp., c) increases. Therefore, we obtain one out of five possible regimes:

1. There exist three steady state points, with $k_{E1} < k_{E2} < k_{E3}$ (this is the specific case portrayed in Figure 1).

2. There exist two steady state points, with $k_{E1} = k_{E2} < k_{E3}$.

3. There exist three steady state points, with $k_{E2} < k_{E1} < k_{E3}$.

4. There exist two steady state points, with $k_{E2} < k_{E1} = k_{E3}$.

5. There exists a unique steady state equilibrium point, corresponding to $E2$.

To assess the stability properties of the steady state(s), consider the Jacobian matrix associated to (16):

$$J(k,q) = \begin{bmatrix} \dfrac{\partial \dot{k}}{\partial k} & \dfrac{\partial \dot{k}}{\partial q} \\ \dfrac{\partial \dot{q}}{\partial k} & \dfrac{\partial \dot{q}}{\partial q} \end{bmatrix} = \begin{bmatrix} f'(k) - \delta & -1 \\ \dfrac{a - c - (N+1)q}{N+1} f''(k) & \delta + \rho - f'(k) \end{bmatrix}.$$

Evaluating J in the Ramsey solution yields.

$$J\left(k^R, q^R\right) = \begin{bmatrix} \rho & -1 \\ \dfrac{a - c - (N+1)\left[f(k^R) - \delta k^R\right]}{N+1} f''(k^R) & 0 \end{bmatrix}.$$

$\left(k^R, q^R\right)$ is a saddle point if $a - c - (N+1)\left[f(k^R) - \delta k^R\right] > 0$. Otherwise, taking the size of the market $a - c = \sigma$ as a bifurcation parameter, we can easily remark that:

- if $\sigma > \max\left\{0, (N+1)\left[\frac{\rho^2}{4f''(k^R)} + f(k^R) - \delta k^R\right]\right\}$, $\left(k^R, q^R\right)$ is an unstable node.
- if $\sigma < (N+1)\left[\frac{\rho^2}{4f''(k^R)} + f(k^R) - \delta k^R\right]$, $\left(k^R, q^R\right)$ is an unstable focus provided that $f(k^R) > \delta k^R - \frac{\rho^2}{4f''(k^R)}$.

Evaluating J in the Cournot-Nash equilibrium, we have:

$$J\left(k^C, q^C\right) = \begin{bmatrix} f'(k^C) - \delta & -1 \\ \\ 0 & \delta + \rho - f'(k^C) \end{bmatrix},$$

whose determinant is negative if $f'(k^C) > \delta + \rho$. This implies that $\left(k^C, q^C\right)$ is a saddle point whenever $q^C < q^R$, while it is an unstable node otherwise.

The discussion carried out so far can be intuitively summarised by noting that the sign of the determinant of the Jacobian matrix is the sign of $a - c - (N+1)\left[f(k^R) - \delta k^R\right] = (N+1)\left(q^C - q^R\right)$ and therefore, if $q^R > q^C$, the saddle point is identified by the intersection of the Cournot-Nash quantity with the locus $\dot{k} = 0$; conversely, if $q^R < q^C$, the saddle point coincides with the Ramsey golden rule. Residually, the dynamics illustrated in Figure 1 intuitively reveals that the origin (point $(0,0)$) is unstable.

The stability analysis reveals that:

Regime 1. $E1$ is a saddle point, while $E2$ is an unstable focus. $E3$ is again a saddle point, with the horizontal line as the stable manifold.

Regime 2. $E1$ coincides with $E2$, so that we have only two steady states which are both saddle points. In $E1 = E2$, the saddle path approaches the saddle point from the left only, while in $E3$ the stable manifold is again the horizontal line.

Regime 3. $E2$ is a saddle, $E1$ is an unstable focus. $E3$ is a saddle point, as in regimes 1 and 2.

Regime 4. Here, $E1$ and $E3$ coincide. $E3$ remains a saddle, while $E1 = E3$ is a saddle whose converging manifold proceeds from the right along the horizontal line.

Regime 5. Here, there exists a unique steady state point, $E2$, which is a saddle point.

We can sum up the above discussion as follows. The unique efficient steady state with saddle point stability is $E2$ if $k_{E2} < k_{E1}$, while it is $E1$ if the opposite inequality holds. Individual equilibrium output is q^C if the equilibrium is in $E1$, or $q^R = f\left(k^R\right) - \delta k^R$ (i.e., the output level corresponding to the optimal capital constraint k^R) if the equilibrium is point $E2$. The reason is that, if the capacity at which marginal instantaneous profit is nil is larger than the optimal capital constraint, the latter becomes binding. Otherwise, the capital constraint is irrelevant, and firms' decisions in each period are driven by the unconstrained maximisation of single-period profits only. Hence, we can state

Proposition 1 *The efficient steady state Nash equilibrium of the open-loop oligopoly game has saddle point stability and is associated to the following individual level of sales*

$$q^N_{OL} = \min\left\{q^C, q^R\right\}.$$

Some additional remarks are in order concerning the inefficient Cournot solution $E3$, whenever such a solution is a saddle point (as in Figure 1). As shown in Cellini and Lambertini (2008), this is a strongly time consistent equilibrium under the open-loop information structure, involving $\lambda_{ii} = 0$, provided the initial capital endowment $k_i(0)$ be large enough to allow the firm to produce q^C in every instant and let the capacity depreciate at the rate δ. If the externality depends on sales, as in this version of the model, adopting this solution has no effect on the amount of pollution. Yet, as we shall see in the remainder, this is no longer true if polluting emissions depend on production or installed capacity.

4 The social optimum

The open-loop Nash solution of the game clearly does not depend on pollution, because of the myopic attitude of firms. In this section we want to establish the conditions under which a social planner that trades-off the negative social externality due to pollution with the profits of the industry and consumer surplus would recommend a lower level of production. Introducing the symmetry assumption, so that $q_i = q$ for all i, the instantaneous social welfare (5) of the social planner is

$$\begin{aligned} sw(t) &= N\pi(t) + \frac{N^2q^2(t)}{2} - \beta N^2q^2(t) \\ &= N(a - c - Nq(t))q(t) + \frac{N^2q^2(t)}{2} - \beta N^2q^2(t) \end{aligned}$$

The social planner aims at maximizing the discounted value of social welfare

$$SW = \int_0^\infty e^{-\rho t} sw(t)dt$$

under the dynamic constraint

$$\dot{k}(t) = f(k(t)) - q(t) - \delta k(t) \tag{20}$$

The current value Hamiltonian for the social planner is (omitting the time argument for brevity)

$$H_{SP} = N(a - Nq - c)q + \frac{N^2q^2}{2} - \beta N^2q^2 + \mu[f(k) - q - \delta k]$$

where μ is the costate variable. The necessary conditions are

$$\frac{\partial H_{SP}}{\partial q} = N\left[a - c - (1+2\beta)Nq\right] - \mu = 0 \tag{21}$$

$$-\frac{\partial H_{SP}}{\partial k} = \mu[\delta - f'(k)] = \dot{\mu} - \rho\mu \Rightarrow \dot{\mu} = \mu\left[\rho + \delta - f'(k))\right] \tag{22}$$

and the transversality condition $lim_{t\to\infty}\mu(t)k(t) = 0$ applies.

From (21) one obtains

$$q = \frac{1}{N^2(1+2\beta)} (aN - cN - \mu) \tag{23}$$

and, differentiating w.r.t. t, we get

$$\dot{q} = -\frac{1}{N^2(1+2\beta)} \dot{\mu}.$$

Using (22) and (21), the latter expression simplifies as

$$\begin{aligned} \dot{q} &= -\frac{1}{N^2(1+2\beta)} (\rho + \delta - f'(k)) \mu \\ &= -\frac{1}{N(1+2\beta)} (\rho + \delta - f'(k)) [a - c - (1+2\beta)Nq] \end{aligned} \tag{24}$$

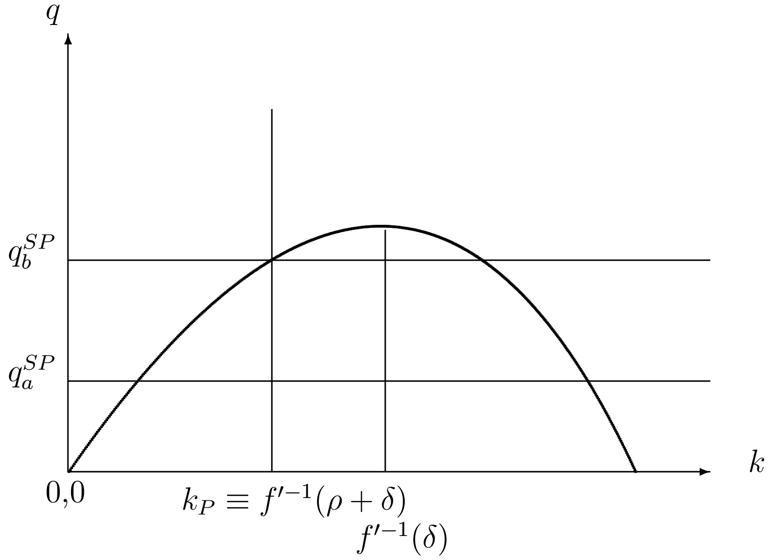
The steady states must satisfy one of the following systems

$$\begin{cases} f'(k^R) = \delta + \rho \\ q^R = f(k^R) - \delta k^R \end{cases}$$

$$\begin{cases} q^{SP} = \dfrac{a-c}{(1+2\beta)N} \\ q^{SP} = f(k^{SP}) - \delta k^{SP} \end{cases}$$

The first solution coincides with the Ramsey golden rule already found in the previous section, while the second solution coincides with that chosen by a social planner in the static case where the market parameters and the sensitivity to pollution are taken into account (to see it, just maximize the instantaneous social welfare function with respect to output q). The two alternative steady states are portrayed in Figure 2.

Figure 2: The phase diagram under social planning



Considering first the market-driven solution, total output is

$$Q^{SP} = Nq^{SP} = \frac{a-c}{1+2\beta}.$$

Comparing the level of total output reached under social planning with total output under Cournot competition, one obtains:

$$Q^{SP} > Q^C \Leftrightarrow \beta < \frac{1}{2N} \equiv \widehat{\beta}.$$

Clearly, in the limit case where $\beta = 0$, one obtains that the level of total output under social planning is necessarily larger than the sales level reached by the industry under Cournot competition. Nevertheless, if the social planner is sensitive to pollution, there is an incentive to reduce the total amount of output. In other words, there is an incentive for the social planner to reduce consumer surplus by decreasing the total amount of output (which corresponds to an increase in prices), as this is more than compensated by the reduction in the amount of pollution and by the increase in total profits. This argument can be reinforced by observing that the industry output under social planning is smaller than under perfect competition $(a - c)$ for all $\beta > 0$. This discussion can be summarised by:

Proposition 2 *Suppose the industry produces* $Q^C = N(a-c)/(N+1)$ *at the Nash equilibrium of the open-loop game, and* $Q^{SP} = (a-c)/(1+2\beta)$. *If so, there exists a threshold value* $\widehat{\beta}$ *above which* $Q^{SP} < Q^C$. *Such a threshold level of* β *is decreasing in* N, *with* $\lim_{N\to\infty}\widehat{\beta} = 0$.

The last remark in the above Proposition entails that, if the market-driven solution prevails under both regimes, an increase in the intensity of market competition is not necessarily welcome from the standpoint of a social planner as it brings about an increase in the total amount of polluting emissions.[4] In the limit, as the Cournot-Nash equilibrium collapses onto perfect competition, any $\beta > 0$ implies that, from the standpoint of the planner, the external effect matters more than the price effect, and therefore the planner produces less than the industry output at the perfectly competitive equilibrium. That is, perfect competition *per se* is not efficient as firms do not internalise the externality.

Now we turn to the alternative case where social planning ends up in the Ramsey equilibrium, which happens when $Q^{SP} > Q^R$ and the latter is a saddle point solution. In this situation, the features of intertemporal capacity accumulation (i.e., parameters ρ and δ and the marginal productivity of capital) matter more than the environmental concern:

Proposition 3 *If* $Q^{SP} > Q^R$, *and therefore the Ramsey golden rule obtains as the socially optimal saddle point equilibrium, the benevolent planner neglects the environmental aspects of the industry and focuses upon optimal intertemporal growth only.*

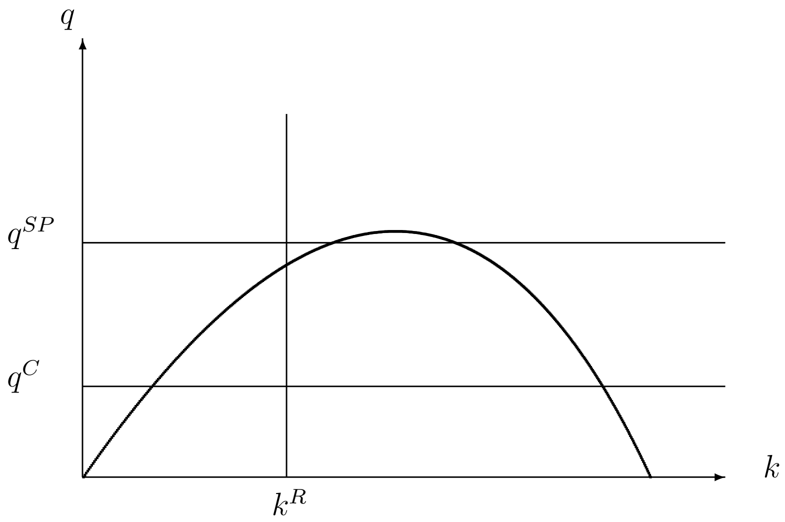
A thorough assessment of the profit-driven equilibrium vs the socially optimal allocation is carried out in next section.

[4]In this respect, a wave of horizontal mergers, or alternatively allowing for some degree of collusion among firms, could be a way of indirectly preserving the environment. To this regard, see Lambertini and Mantovani (2008).

5 Cournot oligopoly vs social planning

To begin with, consider the case where $q^{SP} \geq q^R > q^C$ (as in Figure 3). If so, then the socially optimal allocation reflects the golden rule and the planner neglects the presence of environmental externalities.

Figure 3: $q^{SP} > q^R > q^C$



This is indeed a case where no agent cares about it, as of course profit-seeking firms do not attach any weight to pollution. Hence, this situation is observationally equivalent (at least in terms of the phase diagram and the vector of optimal sales and capital endowment at the steady state(s)) to the case depicted in Cellini and Lambertini (1998, 2008) where environmental effects were ruled out by assumption.

The second case is that where exactly the opposite chain of inequalities applies: $q^C \geq q^R > q^{SP}$ (as in Figure 4). In such a situation, at the Cournot equilibrium the industry produces and sells too much as compared to the social optimum. This may happen if

(i) cost and demand parameters are such that $q^C \geq q^R$, and (ii) β is high enough that $q^R > q^{SP}$.

Figure 4: $q^C > q^R > q^{SP}$

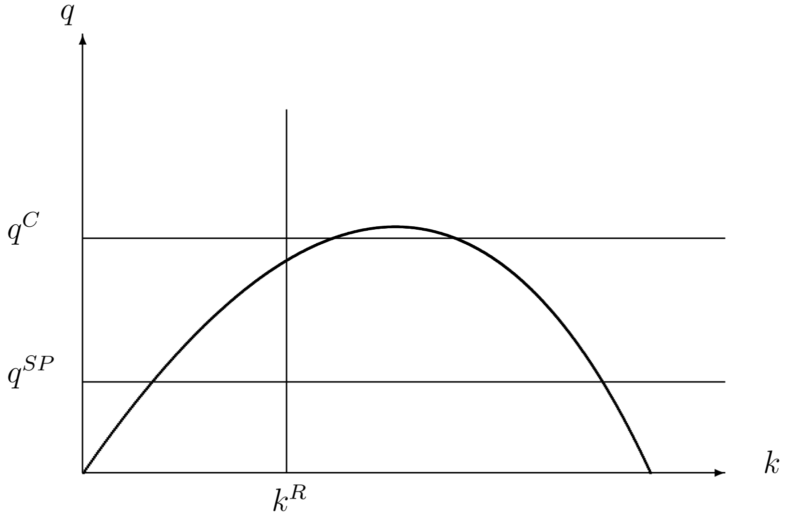


Figure 5: $q^{SP}, q^C > q^R$

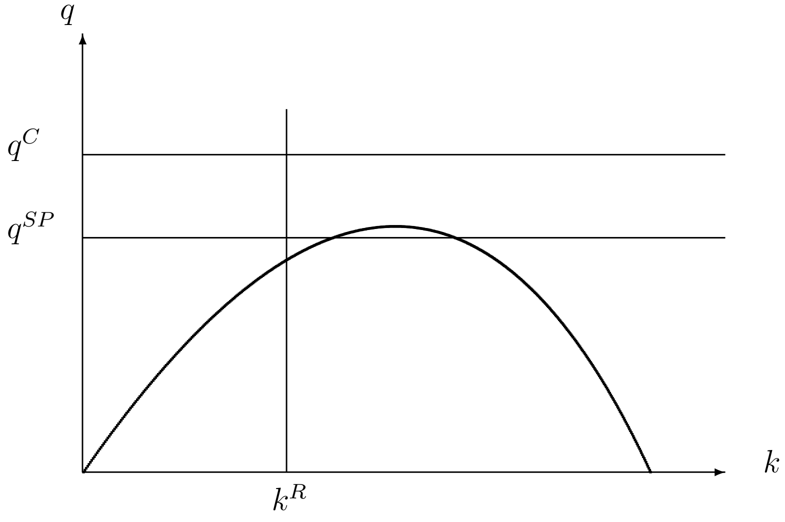
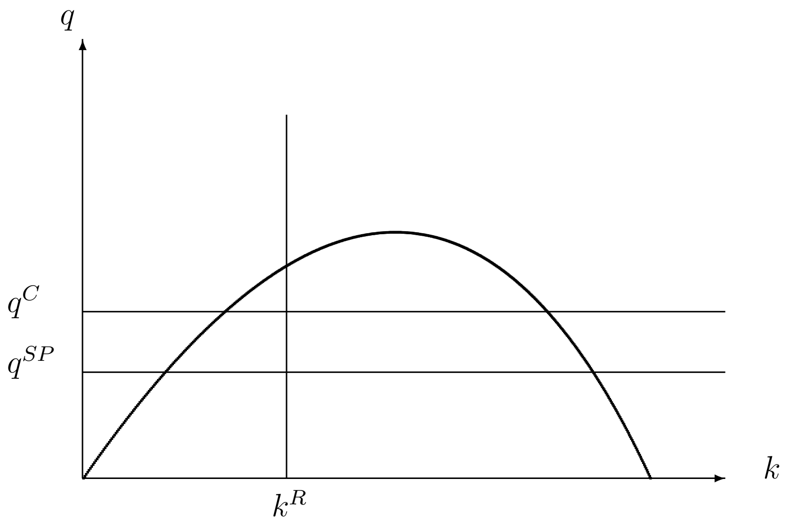


Figure 6: $q^R > q^{SP}, q^C$



In the third case (see Figure 5), $q^C, q^R \geq q^{SP}$ and the Ramsey golden Rule prevails irrespective of the market regime, and once again the steady state allocation is observationally equivalent to the one we would observe without environmental externalities.

Last, there remains the case in which $q^R > q^{SP}, q^C$ (see Figure 6). This is the situation described in Proposition 2, where what matters is the dimension of parameter β.

6 Extension: pollution as a function of production

Here we abandon the assumption that pollution depends quadratically on sales (or consumption), to adopt the alternative view that it is induced by production itself, so that

$$EXT = \beta \left[\sum_{i=1}^{N} f(k_i) \right]^2 . \tag{25}$$

Of course this has no consequences on the behaviour of firms, as they neglect the externality, but it does affect the behaviour of a social planner interested in maximising the discounted flow of social welfare. The planner's Hamiltonian is now:

$$H_{SP} = N(a - Nq - c)q + \frac{N^2 q^2}{2} - \beta N^2 [f(k)]^2 + \mu[f(k) - q - \delta k]$$

The necessary conditions are

$$\frac{\partial H_{SP}}{\partial q} = N[a - c - Nq] - \mu = 0 \tag{26}$$

$$-\frac{\partial H_{SP}}{\partial k} = \mu[\delta - f'(k)] - 2\beta N^2 f(k) f'(k) = \dot{\mu} - \rho\mu \Rightarrow$$

$$\dot{\mu} = \mu[\rho + \delta - f'(k))] - 2\beta N^2 f(k) f'(k) \tag{27}$$

together with the transversality condition $lim_{t\to\infty}\mu(t)k(t) = 0$.

From (26) one obtains

$$\mu = N[a - c - Nq] \tag{28}$$

as well as the control equation:

$$\dot{q} = -\frac{\dot{\mu}}{N^2}. \tag{29}$$

Using (26) and (27), the latter expression simplifies as

$$\begin{aligned} \dot{q} &= -\frac{\mu\left[\rho+\delta-f'(k))\right]-2\beta N^2 f(k)f'(k)}{N^2} \\ &= -\frac{\left(\rho+\delta-f'(k)\right)\left[a-c-Nq\right]-2\beta N f(k)f'(k)}{N} \end{aligned} \tag{30}$$

On the basis of (30), we may state the following:

Lemma 4 *If the environmental externality is determined by the amount of production, then under social planning the industry cannot converge to the Ramsey golden rule for all* $\beta > 0$.

The proof of this claim is intuitive. It suffices to observe that (30) indeed coincides with (24) only in the limit case where $\beta = 0$, but this clearly would imply that the environmental externality is altogether absent.

The stationarity condition $\dot{q} = 0$ admits a unique steady state solution w.r.t. q :

$$q^{SP}(k) = \frac{(a-c)\left[f'(k)-\rho-\delta\right]+2\beta N f(k)f'(k)}{N\left(f'(k)-\rho-\delta\right)}. \tag{31}$$

A sufficient condition for $q^{SP}(k) > 0$ is that $f'(k) > \rho + \delta$.

The Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial q} \\ \frac{\partial \dot{q}}{\partial k} & \frac{\partial \dot{q}}{\partial q} \end{bmatrix} = \begin{bmatrix} f'(k)-\delta & -1 \\ \frac{2\beta N\left[f'(k)\right]^2+\left[a-c-Nq+2\beta N f(k)\right]f''(k)}{N} & \delta+\rho-f'(k) \end{bmatrix},$$

whose trace and determinant are, respectively, $Tr(J) = \rho > 0$ and

$$\Delta(J) = \left[f'(k)-\delta\right]\left[\delta+\rho-f'(k)\right]+\frac{2\beta N\left[f'(k)\right]^2+\left[a-c-Nq+2\beta N f(k)\right]f''(k)}{N}. \tag{32}$$

In correspondence of (31), $\Delta(J)$ simplifies as follows:

$$\Delta(J) = \frac{[\delta - f'(k)][f'(k)-\delta-\rho]^2 - 2\beta\left\{[f'(k)-\delta-\rho][f'(k)]^2 - f(k)(\delta+\rho)f''(k)\right\}}{f'(k)-\delta-\rho}. \tag{33}$$

If $f'(k) > \rho + \delta$, $\Delta(J) < 0$ for all β such that:

$$\beta < \frac{[f'(k)-\delta][f'(k)-\delta-\rho]^2}{2\left\{[f'(k)-\delta-\rho][f'(k)]^2 - f(k)(\delta+\rho)f''(k)\right\}} \tag{34}$$

In such a parameter region, the steady state is stable in the saddle point sense. On the basis of the foregoing discussion, we can formulate

Proposition 5 *If $f'(k) > \rho + \delta$, the steady state solution is a saddle point if the weight attached to pollution in the social welfare function is small enough.*

Alternatively, consider the region where $f'(k) \in (\delta, \rho+\delta)$. Here, $q^{SP} > 0$ if

$$\beta < -\frac{(a-c)[f'(k)-\delta-\rho]}{2Nf(k)f'(k)}. \tag{35}$$

Concerning the sign of $\Delta(J)$, one may easily establish that the sufficient condition for $\Delta(J) > 0$ is that $f(k)(\delta+\rho)f''(k) > [f'(k)-\delta-\rho][f'(k)]^2$. Otherwise, if

$$f(k)(\delta+\rho)f''(k) < [f'(k)-\delta-\rho][f'(k)]^2, \tag{36}$$

then $\Delta(J) < 0$ for all

$$\beta > \frac{[f'(k)-\delta][f'(k)-\delta-\rho]^2}{2\left\{[f'(k)-\delta-\rho][f'(k)]^2 - f(k)(\delta+\rho)f''(k)\right\}}. \tag{37}$$

Hence, whenever $f(k)(\delta+\rho)f''(k) < [f'(k)-\delta-\rho][f'(k)]^2$, any

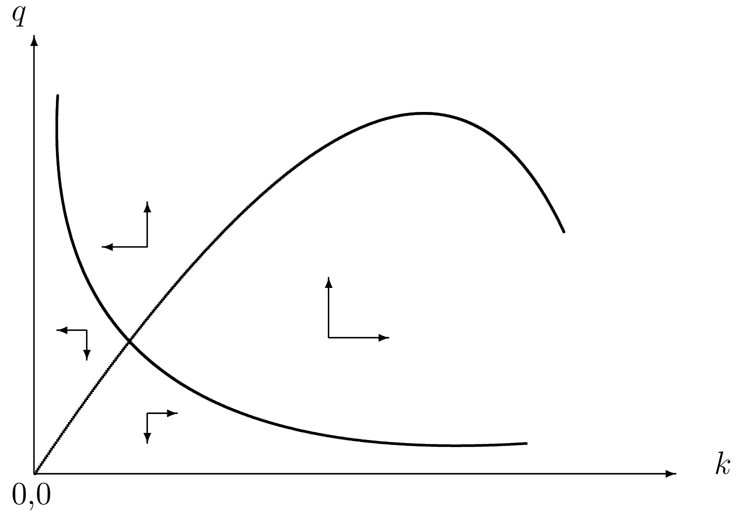
$$\beta \in \left(\frac{[f'(k)-\delta][f'(k)-\delta-\rho]^2}{2\left\{[f'(k)-\delta-\rho][f'(k)]^2 - f(k)(\delta+\rho)f''(k)\right\}}, -\frac{(a-c)[f'(k)-\delta-\rho]}{2Nf(k)f'(k)}\right) \tag{38}$$

ensures that $q^{SP} > 0$ and also entails the saddle point stability. The interval specified in (38) exists if the market size is large enough:

$$a - c > -\frac{Nf(k)[f'(k)-\delta][f'(k)-\delta-\rho]f'(k)}{[f'(k)-\delta-\rho][f'(k)]^2 - f(k)(\delta+\rho)f''(k)} > 0. \tag{39}$$

Note that, if β is to the left of the inf of the interval (38), the equilibrium becomes unstable due to the following mechanism. Suppose the system is in the unstable steady state. The planner knows that a sales expansion induces an increase in consumer surplus and, as a side effect, also a decumulation of capacity and therefore also in production, as $\dot{k}$ becomes negative. This implies a reduction in the externality, which is also desirable. However, in the long run, this deviation is unsustainable as it implies that the size of firms decreases progressively. The phase diagram of this case is illustrated in Figure 7.

Figure 7: The unstable case under planning, with $f'(k) \in (\delta, \rho + \delta)$.



Example As an illustration, assume $f(k) = \alpha\sqrt{k}$, and take

$$a = 2, c = 0, N = 10, \alpha = 1, \delta = 1/20, \rho = 1/18$$

and consider the range where $f'(k) - \delta - \rho > 0$, which entails $k \in (0, 22.438)$. Also, set

$$\begin{aligned}\beta &= \frac{[f'(k)-\delta]\,[f'(k)-\delta-\rho]^2}{2\left\{[f'(k)-\delta-\rho]\,[f'(k)]^2 - f(k)\,(\delta+\rho)\,f''(k)\right\}} - \frac{1}{50} \\ &= \frac{\left(90-19\sqrt{k}\right)^2\left(10-\sqrt{k}\right)}{162000} - \frac{1}{50}\end{aligned}$$

to satisfy (34). Then, impose $\dot{k}=0$ to obtain $q(k) = f(k) - \delta k = \sqrt{k} - k/20$. The equation

$$q^{SP} - q(k) = 0$$

yields the following solutions:

$$k = 0.659,\ k = 24.93 \text{ and } k = 148.996$$

Only the first is acceptable, in view of the above assumption concerning the marginal productivity of capital. In correspondence of $k = 0.659$, the numerical value of the determinant of the Jacobian matrix is $\Delta(J) = -0.018 < 0$, and therefore this qualifies as a saddle point equilibrium. The corresponding optimal steady state quantity is $q^{SP} = 0.779$.

As a last remark, we would like to stress the following. If one keeps in mind that firms disregard the externality and by this very reason are able to converge to the golden rule (under appropriate conditions, which we already know from Section 3), what is likely to appear as the most striking feature of the present version of the model is that the golden rule doesn't look like a socially efficient rule any more because of the fact that pollution is determined by production instead of sales. Consequently, unlike the first version of the model, this one does not allow for any alignment of social and private incentives, except in the very special case in which $\beta = 0$, of course less than interesting as it amounts to assuming that pollution is not there at all.

7 Concluding remarks

We have investigated a dynamic model where an environmental externality interacts with firms' capital accumulation, to show that (i) at the social optimum it may be optimal to trade off some amount of consumer surplus in order to reduce the externality, and (ii) if the external effect is proportional to the industry production, then the Ramsey golden rule just disappears as a stand-alone equilibrium.

The above analysis has been carried out under the open-loop information structure. The desirable extension to feedback models is left for future research.

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