

Process and Product Innovation: a Differential Game Approach to Product Life Cycle¹

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December 18, 2009

¹This paper is part of the second author's PhD dissertation at CORE, Université Catholique de Louvain. We thank Makoto Yano (Editor), an anonymous referee, Rabah Amir, Ramon Casadesus, Bruno Cassiman, Roberto Cellini, Claude d'Aspremont, Arsen Palestini, Jacques-François Thisse, Giovanni Valentini, Vincent Vannetelbosch and the audience at Bologna, Barcellona (IESE), IFORS 2005 (Waikiki), EARIE 2005 (Oporto), and Institute Research Workshop 2005 (Ferrara) for precious comments and discussion. The usual disclaimer applies.

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Abstract

We investigate the timing of adoption of product and process innovation using a differential game where firms may invest in both activities. We consider horizontal product innovation that reduces product substitutability, and process innovation that reduces marginal cost. First, we demonstrate that the incentive for cost-reducing investment is relatively higher than the incentive to increase product differentiation. Second, depending on initial conditions, (i) firms activate both types of investment from the very outset to the steady state; (ii) firms initially invest only in one R&D activity and then reach the steady state either carrying out only such activity or carrying out both; (iii) firms do not invest at all in either type of innovation. Comparing R&D investments under Cournot and Bertrand behaviour shows that quantity competition entails lower R&D incentives than price competition in both directions.

Running head: Process and product innovation

JEL classification: C73, D43, D92, O31.

Keywords: product innovation, process innovation, differential games, technology life cycle.

1 Introduction

Casual observation tells that firms carry out both product and process innovations, these being driven either by technological complementarities or by the life cycle of technology. However, the bearings of these two factors on R&D efforts have been investigated separately.

One stream of research has focussed upon complementarities within R&D portfolios in monopoly (Athey and Schmutzler 1995; Lambertini and Orsini 2000; Lambertini 2003, 2004; Lin 2004) or oligopoly (Rosenkranz 1995; Bonanno and Haworth 1998; Lin and Saggi 2002). The bottom line of this stream of research is that R&D efforts in each direction boosts the firms' incentive to carry out analogous effort in the other direction.

On the other hand, the technology life cycle describes the dynamics of product and process innovation. The common view is that product innovation necessarily precedes process innovation. Many reasons have been adduced to explain such belief. Abernathy and Utterback (1975, 1982) propose a model where firms initially tend to direct most of their R&D resources to product innovation, because the market potential is large and products have to find a proper "dominant design" (Porter 1983) to match consumers' requirements. As the proliferation of brand new products crowds the market, R&D profitability fades away and firms shift from product to process innovation. Klepper (1996; see also Filson 2002) highlights the role of firms' innovative capabilities and size in conditioning R&D spending. In particular, he suggests that the returns to product innovation are highest at the very beginning because they depend on the acquisition of new consumers, while the returns to process innovation are very attractive at a later stage because they are proportional to the level of output produced by the firm. This gives a rationale for the fact that firms tend to pursue first product and then process innovation.

Adner and Levinthal (2001) move some critiques to the traditional approach to technology life cycle, that they call a "supply-side" view of technological change. The main

limitation they emphasize is the exclusive attention to the internal capabilities of the firm. They propose instead a “demand side” approach where technology changes are driven by the interaction between technology development and consumers’ heterogeneous demands. Unlike what the traditional view maintains, they claim that the first phase is not always characterized by the adoption of product innovation. This will depend on the initial performance of the product on the market. The second phase is still characterized by the predominance of process innovation, but it does not necessarily imply a sharp decrease in the product innovation component. In addition to that, it allows for the possibility that there exists a third stage, not treated formally in the previous literature, where innovative activity is a balance of product and process R&D in order to keep the price relatively stable and to increase at the same time the relative performance of the product. Adner and Levinthal’s distinction between a technology that is “new to the world” and one that is “new to the market” is crucial to understand which activity firms implement first. The former corresponds to a product whose initial technology is “crude”. The product is substantially new for customers and an investment in product innovation is appealing due to the potential market that could be created or expanded. In this situation the conventional life cycle holds, with an initial predominance of product innovation and a subsequent shift towards process innovation. The latter refers to a product whose technology is very advanced; the possibilities for improving the functionality of the product are then limited and firms may only invest to reach a larger share of consumers. Initially they find it profitable to implement process innovation to lower the cost of production but after a certain point they switch to product innovation to further expand the market.

Others have stressed the difference between innovations and improvements, i.e., between technological breakthroughs and engineering refinements (see Doraszelski 2004). This allows one to understand that firms may not wait for future breakthroughs, while there exists an incentive for them to delay the adoption of a new technology until it is sufficiently developed.

This paper bridges the two aforementioned lines of research and investigates the technology life cycle by using a differential game in which firms invest either in process innovation or in product innovation, or in both. In our framework product innovation is conceived in a “horizontal” way as a reduction of product substitutability. Both investment activities generate a positive spillover for the rival.

The setup we adopt in the remainder of the paper allows one to analyze both the innovation portfolio decisions and the life cycle characterizing innovative activities. In the light of the conclusions drawn by Adner and Levinthal (2001), and by virtue of the dynamic nature of our approach, we shall privilege the latter aspect rather than the former. In line of principle, there exist four different perspectives: (i) firms activate both types of investment activities from the outset up to the steady state; (ii) firms initially invest only in product differentiation and then reach the steady state either carrying out only product innovation or carrying out both; (iii) firms initially invest only to reduce the cost of production and then arrive at the steady state either carrying out only process innovation or carrying out both; (iv) firms do not invest at all in either type of innovation, and they are driven to the steady state by the depreciation rate.

We set out by modelling a Cournot market where firms may invest in either activity alternatively, or in both of them at the same time. The main results stemming from this setup are twofold. First, we demonstrate that the incentive for cost-reducing investment is relatively higher than the incentive to increase product differentiation. This is due to the fact that, in the present setup, product differentiation is a public good, in that any investment in this direction by either firm fully spills over to the rival via the demand function.¹ Second, we show that the implementation of both innovative activities depends on the initial degree of product substitutability compared with the initial marginal cost

¹This is consistent with the conclusions drawn by Cohen and Klepper (1996) in a different model. They find that, as long as the marginal costs of the two R&D activities are comparable, an increase in firms’ output has a bigger proportionate effect on process than product R&D.

of production. In particular, this is surely the case if initial conditions tell that products are very close (or even perfect) substitutes and marginal cost is very close to consumers' reservation price. Whenever both innovative activities are jointly implemented, they exhibit complementarity, as shown in previous research on static models of innovation (Lin and Saggi 2002; Lambertini 2003). The differential game approach allows us to highlight the existence of complementarity as an inherent feature of dynamic optimization.

Then, we briefly treat the Bertrand case, whose technical features are largely the same as in the Cournot model, except that the presence of price competition clearly exerts different effects on firms' incentives to carry out both types of R&D. Specifically, we find that Bertrand behavior drives higher R&D efforts both in process and in product innovation. This outcome has a definite Arrowian interpretation, as strengthening market competition ultimately involves a faster pace of technical progress.

The multiplicity of equilibrium paths generated by our model is confirmed by casual observation. The standard case where product innovation precedes process innovation is, e.g., that of CD players or digital cameras, where the new product has been introduced into the market at high prices, and subsequently the abatement of production costs has allowed firms to supply comparable varieties at lower prices. Conversely, the opposite happens when firms restyling existing products once they have already abated production costs as much as possible. Examples of this can be found in the car industry, for instance this applies for utility cars like Fiat Panda and Lancia Y10, which have undergone a typical process of horizontal differentiation (i.e., restyling) at an advanced stage of their life cycle. Finally, the third type of equilibrium with both innovations taking place at the same time is observed in the PC industry (see Adner and Levinthal 2001; Adner 2004).

The outline of the paper is as follows. Section 2 illustrates the basics of the model. Section 3 analyses the Cournot game, treating the two cases where firms perform only one type of activity, as well as that in which firms invest both in process and in product innovation, and provides the main results of the paper concerning the time of adoption.

Section 5 describes the joint adoption of product and process innovation under Bertrand competition. Section 6 concludes the paper.

2 The Model

We build a differential duopoly game over continuous time $t \in [0, \infty)$, where at any instant firms choose the quantity level and the investment level in either product or process innovation, or both. Process innovation is formalized as a reduction in the unit cost of production, while product innovation affects product substitutability as perceived by consumers.

Firms produce differentiated goods. As in Spence (1976) and Singh and Vives (1984), the representative consumer's instantaneous utility function is defined in terms of the consumption of the two differentiated goods and the numeraire good m , and is given by:

$$u(q_1(t), q_2(t), m) = a(q_1(t) + q_2(t)) - ((q_1(t))^2 + (q_2(t))^2)/2 - s(t)q_1(t)q_2(t) + m, \quad (1)$$

where $s(t) \in [0, 1]$. Constrained utility maximization for any given price pair $\{p_i(t), p_j(t)\}$ gives rise to the following inverse demand system, to be used under Cournot competition:

$$p_i(t) = a - q_i(t) - s(t)q_j(t) \quad \forall i \neq j, \quad i, j = \{1, 2\} \quad (2)$$

where the parameter $s(t)$ represents the degree of substitutability between the two products. If $s(t) = 1$ products are completely homogeneous. At the opposite, for $s(t) = 0$, products are independent and each firm acts as a monopolist. By inverting the system (2), one obtains the direct demand for the Bertrand case:

$$q_i(t) = \frac{a}{1 + s(t)} - \frac{p_i(t)}{1 - [s(t)]^2} + \frac{s(t)p_j(t)}{1 - [s(t)]^2}. \quad (3)$$

At any time t , the output level $q_i(t)$ is produced at a constant marginal cost. Accordingly, the instantaneous cost function is $C_i(c_i, q_i) = c_i(t)q_i(t)$.

We assume that product differentiation can be affected by firms' R&D investment in product innovation. The dynamics associated to product innovation is described by the kinematic equation:

$$\frac{ds(t)}{dt} \equiv \dot{s} = s(t) [-x_i(t) - x_j(t) + \delta] \quad (4)$$

where $x_i(t)$ represents the amount of effort made by firm i at time t in order to increase product differentiation through a reduction of $s(t)$.² The parameter $\delta \in [0, 1]$ indicates the depreciation rate due to ageing of technology, which is common to both firms and constant over time. A characteristic associated with horizontal product differentiation is that it gives rise to a complete spillover effect between the R&D activity undertaken by the firms.³ Equation (4) can be rewritten as follows:

$$\frac{\dot{s}}{s(t)} = -x_i(t) - x_j(t) + \delta \quad (5)$$

so as to highlight that the rate of change of product substitutability over time is linear in the instantaneous investment efforts.

Moreover, firms may invest in process innovation; as a consequence, the marginal cost borne by firm i evolves over time as described by the following kinematic equation:

$$\frac{dc_i(t)}{dt} \equiv \dot{c}_i = c_i(t) [-k_i(t) - \theta k_j(t) + \eta] \quad (6)$$

where $k_i(t)$ indicates the effort made by firm i to reduce the cost of production. Parameter $\theta \in [0, 1]$ measures the positive technological spillover that firm i receives from the process innovative activity of the rival, while $\eta \in [0, 1]$ is the depreciation rate, assumed to be

²The idea that s depends upon firms' investment decisions has been investigated in static models by Harrington (1995), Lambertini and Rossini (1998) and Lambertini et al. (1998). Recent contributions apply this idea to differential games, e.g. Cellini and Lambertini (2002, 2004).

³Given the demand setup we adopt here, the existence of a full externality in product innovation is straightforward because $s(t)$ is a common state. Yet, this feature would also appear under the address approach (e.g., *à la* Hotelling 1929) since any relocation of a firm's product away from the rival's indeed benefits both competitors.

common to both firms and constant over time. That is, η indicates that the production technology for the final good becomes more and more obsolete if firms do not invest in process R&D. Bluntly, producing with old machines is costlier than with machines incorporating a steady technical progress. It is easy to notice that the rate of change of firm i 's marginal cost over time is linear in the instantaneous investment efforts, given that:

$$\frac{\dot{c}_i}{c(t)} = -k_i(t) - \theta k_j(t) + \eta. \quad (7)$$

Notice that (6) is indeed a dynamic version of the linear R&D technology employed by d'Aspremont and Jacquemin (1988).

The instantaneous cost of investing in product innovation and in process innovation is respectively given by $C_i(x_i(t)) = \gamma [x_i(t)]^2$ and $Z_i(k_i(t)) = \beta [k_i(t)]^2$, where γ and β are positive parameters. To firm i , hence, investing in both types of R&D implies decreasing returns to innovative activity. The parameters γ and β are inverse measures of product and process innovation R&D efficiency, respectively.

Instantaneous profits are given by:

$$\pi_i(t) = [p_i(t) - c_i(t)] q_i(t) - \gamma [x_i(t)]^2 - \beta [k_i(t)]^2 \quad (8)$$

We assume that firm i aims at maximizing the discounted profit flow:

$$\Pi_i(t) = \int_0^{\infty} \pi_i(t) e^{-\rho t} dt \quad (9)$$

w.r.t. controls $x_i(t)$, $k_i(t)$ and the relevant market variable (either $q_i(t)$ or $p_i(t)$), under the constraints given by the state dynamics (4) and (6). The discount rate $\rho > 0$ is assumed to be constant and common to both firms.

If both types of R&D activity are carried out, the corresponding current value Hamiltonian function is:

$$\mathcal{H}_i(t) = e^{-\rho t} \left[\pi_i(t) + \lambda_i(t) \dot{s} + \lambda_{ii}(t) \dot{c}_i + \lambda_{ij}(t) \dot{c}_j \right] \quad (10)$$

where $\lambda_i(t) = \mu_i(t)e^{\rho t}$, $\lambda_{ii}(t) = \mu_{ii}(t)e^{\rho t}$ and $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$, μ_i , $\mu_{ii}(t)$ and $\mu_{ij}(t)$ being the co-state variable associated to $s(t)$, $c_i(t)$ and $c_j(t)$, respectively.

3 The Cournot Game

Here we assume firms use quantities as the market variables. In order to shed light on the relation between product and process innovation, we start by considering the two activities separately. Then, we will consider the case where firms implement both types of innovative activity.

3.1 Product Innovation

Firms invest only in product innovation aiming at reducing the degree of product substitutability. At this time process innovation is not taken into account, hence $k_i(t) = 0$ and the marginal cost c does not depend on firms' innovative activity. Without loss of generality, we can assume that the marginal cost is constant and common for both firms, hence $c_i = c$. Instantaneous profits are given by:

$$\pi_i(t) = [a - c - q_i(t) - s(t)q_j(t)] q_i(t) - \gamma [x_i(t)]^2 \quad (11)$$

Firm i maximizes the discounted profit flow $\Pi_i(t) = \int_0^\infty \pi_i(t) e^{-\rho t} dt$ w.r.t. controls $x_i(t)$ and $q_i(t)$, under the only constraint given by (4). The corresponding Hamiltonian function writes:

$$\begin{aligned} \mathcal{H}_i(t) = & e^{-\rho t} \{ [a - c] q_i(t) - q_i(t)^2 - s(t)q_i(t)q_j(t) - \gamma [x_i(t)]^2 + \\ & + \lambda_i(t)s(t) [-x_i(t) - x_j(t) + \delta] \}. \end{aligned} \quad (12)$$

The initial condition is $s(0) = s_0 \in (0, 1]$. Firms play simultaneously. Firm i 's first

order conditions (FOCs) on controls are:⁴

$$\frac{\partial \mathcal{H}_i(t)}{\partial q_i(t)} = a - c - 2q_i(t) - s(t)q_j(t) = 0 \Rightarrow q_i^*(t) = \frac{a - c - s(t)q_j(t)}{2} \quad (13)$$

$$\frac{\partial \mathcal{H}_i(t)}{\partial x_i(t)} = -2\gamma x_i(t) - \lambda_i(t)s(t) = 0 \Rightarrow x_i^*(t) = -\frac{s(t)\lambda_i(t)}{2\gamma} \quad (14)$$

Note that both (13) and (14) contain the state variable $s(t)$, which is common for both firms. As a consequence, the open-loop solution and the closed-loop one do not coincide. The solution concept we adopt is the closed-loop Nash equilibrium. We then take into account the feedback between player i 's strategy and player j 's state variable. This will lead to a Markovian (strongly time consistent) equilibrium characterized by subgame perfection.

We specify firm i 's co-state equation containing the feedback effects:

$$\begin{aligned} -\frac{\partial \mathcal{H}_i(t)}{\partial s(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial q_j(t)} \frac{\partial q_j^*(t)}{\partial s(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial x_j(t)} \frac{\partial x_j^*(t)}{\partial s(t)} &= \dot{\lambda}_i - \rho \lambda_i(t) \Rightarrow \\ \dot{\lambda}_i &= q_i(t)q_j(t) - \frac{s(t)[q_i(t)]^2}{2} - \frac{\lambda_i(t)\lambda_j(t)s(t)}{2\gamma} + \lambda_i(t)[- \delta + x_i(t) + x_j(t) + \rho] \end{aligned} \quad (15)$$

along with the transversality condition:

$$\lim_{t \rightarrow \infty} \mu_i(t) s(t) = 0. \quad (16)$$

For the sake of simplicity, we introduce the following:⁵

Assumption $q_i(t) = q_j(t) = q(t)$ and $x_i(t) = x_j(t) = x(t)$.

⁴For the sake of brevity, in the remainder we omit the indication of exponential discounting. Moreover, second order conditions are also omitted throughout the paper. They are always met, as it can be verified using different methods (see Mangasarian 1966; Arrow 1968; Stalford and Leitmann 1973).

⁵It is known that continuous time models can indeed yield multiple equilibria. To this regard, see Clemhout and Wan (1993), *inter alia*.

Then, from (13), we derive the equilibrium per firm output

$$q(t) = \frac{a - c}{2 + s(t)}, \quad (17)$$

which coincides with the standard outcome of Cournot models with product differentiation (see Singh and Vives 1984; Cellini and Lambertini 1998). Hence, given the implied symmetry condition $\lambda_i(t) = \lambda_j(t) = \lambda(t)$, from (14) we know that:

$$-\lambda(t) = \frac{2\gamma x(t)}{s(t)} \quad (18)$$

By symmetry, and using (18), the co-state equation (15) simplifies as follows:

$$\dot{\lambda} = \frac{[2 - s(t)] [q(t)]^2 - 4\gamma x(t) [3x(t) + \rho - \delta]}{2s(t)} \quad (19)$$

From (18) we obtain $x(t)$, which can be differentiated w.r.t. t :

$$\dot{x} = \frac{1}{2\gamma} \left[-\dot{\lambda} s(t) - \lambda(t) \dot{s} \right] \quad (20)$$

where, following the symmetry assumptions,

$$\dot{s} = s(t) [-2x(t) + \delta] \quad (21)$$

Then, plugging (19), (18) and (21) into (20) and rearranging, one obtains:

$$\dot{x} = x(\rho + x) - \frac{s(t)(a - c)^2 [2 - s(t)]}{4\gamma [2 + s(t)]^2}. \quad (22)$$

The expression on the r.h.s. of (22) is valid for all $s(t) \in (0, 1]$; if instead $s(t) = 0$, optimal per period investment is $x(t) = 0$.

We are now in a position to assess the overall dynamic properties of the game, being it fully characterized by (22) and (21). Steady state solutions are obtained by solving the system $\dot{s} = 0$ and $\dot{x} = 0$. Given that explicit solutions are cumbersome, we start by imposing $\dot{x} = 0$ to determine an equilibrium relation between $x(t)$ and $s(t)$:

$$x(s) = \frac{-\gamma\rho [2 + s(t)] \pm \sqrt{\gamma \{ \gamma\rho^2 [2 + s(t)]^2 + (a - c)^2 [2 - s(t)] s(t) \}}}{2\gamma [2 + s(t)]} \quad (23)$$

The smaller root corresponds to a locus where x has always a negative value, and can therefore be disregarded, being economically meaningless. Therefore, the optimal investment is:

$$x^{Pd}(s) = \frac{-\gamma\rho[2 + s(t)] + \sqrt{\gamma\{\gamma\rho^2[2 + s(t)]^2 + (a - c)^2[2 - s(t)]s(t)\}}}{2\gamma[2 + s(t)]} \quad (24)$$

where the superscript Pd stands for *product innovation*. Hence, by substituting (24) in (4) and solving for s , we get three solutions:

$$s = 0; \quad s = \frac{(a - c) \left[(a - c) \pm \sqrt{(a - c)^2 - 8\delta\gamma(\delta + 2\rho)} \right] - 2\delta\gamma(\delta + 2\rho)}{(a - c)^2 + \delta\gamma(\delta + 2\rho)} \quad (25)$$

It is immediate to prove that:

Lemma 1 The solutions $s = \frac{(a - c) \left[(a - c) \pm \sqrt{(a - c)^2 - 8\delta\gamma(\delta + 2\rho)} \right] - 2\delta\gamma(\delta + 2\rho)}{(a - c)^2 + \delta\gamma(\delta + 2\rho)}$ are real if and only if $(a - c)^2 - 8\delta\gamma(\delta + 2\rho) > 0$, i.e. $c \in \left[0, a - 2\sqrt{2\delta\gamma(\delta + 2\rho)} \right]$. This also ensures that $s \in [0, 1]$.

The statement of the above lemma amounts to requiring that the marginal cost of producing the final good be sufficiently low. Alternatively, the above solutions are real iff $\gamma \leq \frac{(a - c)^2}{8\delta(\delta + 2\rho)}$, i.e., for sufficiently low values of the marginal cost of the R&D activity. Both interpretations boil down to the idea that market profitability (measured by $(a - c)^2$) must be large enough as compared to the cost of R&D activity in order for the model to generate an economically meaningful steady state. If this condition (either in the former or in the latter form) is not satisfied, then the associated R&D effort is $x^{Pd} < \delta/2$ and therefore $\dot{s} > 0$ forever. We will come back to this point while analyzing the possibility for firms to undertake simultaneously process and product R&D activities.

Considering the stability of the system, the following can be shown to hold:

Proposition 1 *Provided that $c \in [0, a - 2\sqrt{2\delta\gamma(\delta + 2\rho)}]$, the steady state point*

$$\begin{aligned} s^{Pd} &= \frac{(a - c) \left[(a - c) - \sqrt{(a - c)^2 - 8\delta\gamma(\delta + 2\rho)} \right] - 2\delta\gamma(\delta + 2\rho)}{(a - c)^2 + \delta\gamma(\delta + 2\rho)}, \\ x^{Pd} &= \frac{\delta}{2} \end{aligned}$$

is the unique saddle point equilibrium of the game when firms invest only in product innovation.

Proof. The Jacobian matrix of the dynamic system formed by (4) and (22) is (under symmetry):

$$J^{Pd} = \begin{bmatrix} \frac{\partial \dot{s}}{\partial s} = \delta - 2x & \frac{\partial \dot{s}}{\partial x} = -2s \\ \frac{\partial \dot{x}}{\partial s} = \frac{(a - c)^2 (3s - 2)}{2\gamma(2 + s)^3} & \frac{\partial \dot{x}}{\partial x} = \rho + 2x \end{bmatrix} \quad (26)$$

whose trace and determinant are:

$$\begin{aligned} T(J^{Pd}) &= \rho + \delta; \\ \Delta(J^{Pd}) &= (\delta - 2x)(\rho + 2x) + \frac{(a - c)^2 (3s - 2)s}{\gamma(2 + s)^3}. \end{aligned} \quad (27)$$

Evaluating $\Delta(J^{Pd})$ in (s^{Pd}, x^{Pd}) , we have:

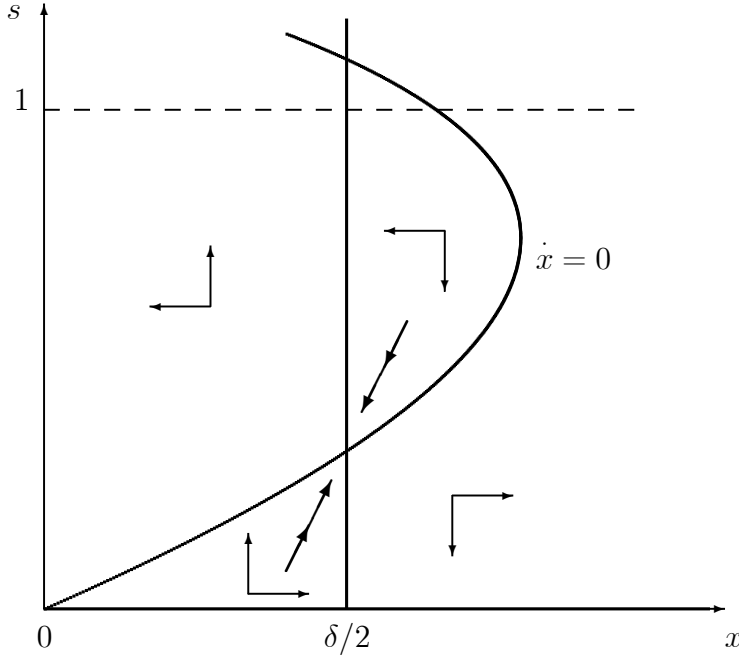
$$\Delta(J^{Pd}) \propto -16\delta\gamma(\delta + 2\rho) [(a - c)^2 - 8\delta\gamma(\delta + 2\rho)] [(a - c)^2 + \delta\gamma(\delta + 2\rho)] \quad (28)$$

where the expression on the r.h.s. is surely negative on the basis of the condition warranting the reality of s^{Pd} . Therefore, $\Delta(J^{Pd}) < 0$ in (s^{Pd}, x^{Pd}) , which qualifies as the unique saddle point equilibrium. ■

An intuitive illustration of saddle point stability can be obtained by investigating the dynamics of the system in the positive quadrant of the space $\{s, x\}$, which is described in Figure 1. The locus $\dot{s} = 0$ is given by the $s = 0$ and $x = \delta/2$, while the locus $\dot{x} = 0$

draws a curve over the admissible values of s .⁶ The dynamics of s and x are respectively summarized by vertical and horizontal arrows.

Figure 1 : The phase diagram in the product innovation case



From the phase diagram, it is clear that this saddle point equilibrium can be approached only along the northeast arm of the saddle path. It is also immediate to demonstrate that $s^{Pd} \in [0, 1]$.

Comparative statics exercises can be carried out on $x^{Pd}(s)$ to show that:

$$\frac{\partial x^{Pd}(s)}{\partial s} \propto 2 - 3s(t) ; \frac{\partial x^{Pd}(s)}{\partial c} < 0. \quad (29)$$

The first partial derivative implies that the incentive to implement product innovation is non-monotone in the degree of product differentiation. In particular, $\partial x^{Pd}(s)/\partial s > 0$ if and only if $s \in [0, 2/3)$, i.e., firms have a strong incentive to differentiate their products

⁶The curve $\dot{x} = 0$ may or may not cross twice the vertical line $x = \delta/2$ within the admissible values of $s \in [0, 1]$. In Figure 1 we depict the situation where they cross only once.

varieties if they are close substitutes. As soon as product differentiation is large enough, R&D efforts start decreasing. The second derivative in (29) shows that the optimal investment is everywhere decreasing in the level of marginal cost, for any given $s(t)$ before the steady state is reached. The obvious reason is that a decrease in c , all else equal, entails a higher gross profits and therefore a larger source of internal funds to be invested in R&D activities.

Now examine the effect of a variation in c on the steady state level of substitutability (see Proposition 1), to verify easily that $\partial s^{Pd}/\partial c > 0$ always. The explanation is that, the higher is c , the lower is the amount of internal funds available to finance R&D at any time during the game; as a result, the steady state degree of substitutability is positively related to marginal production cost. The same applies along the optimal investment path, as it can be verified by inverting $x^{Pd}(s)$ to obtain $s^{Pd}(x)$, and then checking that $\partial s^{Pd}(x)/\partial c > 0$.

3.2 Process Innovation

Consider now the case where firms undertake only R&D for process innovation, hence $x_i(t) = 0$ and the parameter s measuring product substitutability does not change over time. Instantaneous profits are given by:

$$\pi_i(t) = [a - c_i(t) - q_i(t) - sq_j(t)] q_i(t) - \beta [k_i(t)]^2 \quad (30)$$

Firms maximize the discounted profit flow w.r.t. controls $k_i(t)$ and $q_i(t)$, under the constraints (6). The corresponding Hamiltonian function writes:

$$\begin{aligned} \mathcal{H}_i(t) = & e^{-\rho t} \{ [a - c(t)] q_i(t) - q_i(t)^2 - sq_i(t)q_j(t) - \beta [k_i(t)]^2 + \\ & + \lambda_{ii}(t)c_i(t) [-k_i(t) - \theta k_j(t) + \eta] + \lambda_{ij}(t)c_j(t) [-k_j(t) - \theta k_i(t) + \eta] \}. \end{aligned} \quad (31)$$

The initial condition is $c(0) = c_0 \in (0, a]$. Firm i 's first order conditions (FOCs) on

controls are:⁷

$$\frac{\partial \mathcal{H}_i(t)}{\partial q_i(t)} = a - c_i(t) - 2q_i(t) - sq_j(t) = 0 \Rightarrow q_i^*(t) = \frac{a - c_i(t) - sq_j(t)}{2} \quad (32)$$

$$\frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} = -2\beta k_i(t) - \lambda_{ii}(t)c_i(t) - \lambda_{ij}(t)\theta c_j(t) = 0 \Rightarrow k_i^*(t) = -\frac{[c_i(t)\lambda_{ii}(t) + \theta\lambda_{ij}(t)c_j(t)]}{2\beta} \quad (33)$$

Note that (32) does not contain the state variable (6) of the rival, which is however present in (33). At first sight, there seem to be a feedback between the R&D decisions, at least for any positive spillover effect.⁸ However, it is possible to prove that:

Lemma 2 *The open-loop Nash equilibrium of the game with firms investing only in process innovation is subgame (or Markov) perfect.*

Proof. We show that the present game is a *perfect game* in the sense of Leitmann and Schmitendorf (1978) and Feichtinger (1983).⁹ We basically need to verify that no feedback effects are actually present, even for positive spillover levels. The following proof follows Cellini and Lambertini (2005).

Let us write the co-state equations:

$$-\frac{\partial \mathcal{H}_i(t)}{\partial c_i(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial k_j(t)} \frac{\partial k_j^*(t)}{\partial c_i(t)} = \dot{\lambda}_{ii} - \rho\lambda_{ii}(t) \Leftrightarrow \quad (34)$$

$$\dot{\lambda}_{ii} = q_i(t) + \lambda_{ii}(t) [k_i(t) + \theta k_j(t) + \rho - \eta] - \frac{\theta}{2\beta} \lambda_{ji}(t) [\lambda_{ij}(t)c_j(t) + \theta\lambda_{ii}(t)c_i(t)]$$

$$-\frac{\partial \mathcal{H}_i(t)}{\partial c_j(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} \frac{\partial k_i^*(t)}{\partial c_j(t)} = \dot{\lambda}_{ij} - \rho\lambda_{ij}(t) \Leftrightarrow \quad (35)$$

⁷As before, we omit the indication of the exponential discounting.

⁸If $\theta = 0$ the two investment plans are completely independent and therefore it is evident that no feedback effect operates.

⁹A differential game is *perfect* whenever the closed-loop equilibrium collapses into the open-loop one, the latter being thus strongly time consistent, i.e., subgame perfect. For further details, see Mehlmann (1988, Ch. 4) and Dockner et al. (2000, Ch. 7).

$$\dot{\lambda}_{ij} = \lambda_{ij}(t) \left\{ [k_j(t) + \theta k_i(t) + \rho - \eta] - \frac{\theta}{2\beta} [2\beta k_i(t) + \lambda_{ii}(t)c_i(t) + \theta \lambda_{ij}(t)c_j(t)] \right\}$$

where

$$\frac{\partial \mathcal{H}_i(t)}{\partial k_j(t)} \frac{\partial k_j^*(t)}{\partial c_i(t)}$$

captures the feedback effects, and partial derivatives $\partial k_j^*(t)/\partial c_i(t)$ are calculated using the optimal values of investment as from (33), i.e.

$$k_j^*(t) = -\frac{c_j(t)\lambda_{jj}(t) + \lambda_{ji}(t)\theta c_i(t)}{2\beta} \quad (36)$$

These conditions must be evaluated along with the initial conditions $\{c_i(0)\} = \{c_{0,i}\}$ and the transversality conditions

$$\lim_{t \rightarrow \infty} \mu_{ij}(t) c_j(t) = 0, \quad i, j = 1, 2. \quad (37)$$

Now note that (35) is a differential equation in separable variables admitting the solution $\lambda_{ij}(t) = 0$ at all times. Accordingly, we can rewrite the expression for the optimal investment of firm i as follows:

$$k_i^*(t) = -\frac{\lambda_{ii}(t)c_i(t)}{2\beta} \quad (38)$$

which entails that $\partial k_i^*(t)/\partial c_j(t) = 0$. This holds for both firms due to the underlying symmetry of the model. Feedback (cross-) effects are nil along the equilibrium path and hence the open-loop is a degenerate closed-loop one, and it is strongly time consistent, or equivalently, subgame perfect.¹⁰ ■

On the basis of Lemma 2, we can proceed with the characterization of the open-loop solution. The first order conditions on control as well as the transversality conditions are the same as above, while the co-state equations simplify as follows:

$$-\frac{\partial \mathcal{H}_i(t)}{\partial c_i(t)} = \dot{\lambda}_{ii} - \rho \lambda_{ii}(t) \Leftrightarrow \dot{\lambda}_{ii} = q_i(t) + \lambda_{ii}(t) [k_i(t) + \theta k_j(t) + \rho - \eta] \quad (39)$$

¹⁰For a dynamic R&D game where knowledge accumulation has a strategic effect during the game itself, see Doraszelski (2003).

$$-\frac{\partial \mathcal{H}_i(t)}{\partial c_j(t)} = \dot{\lambda}_{ij} - \rho \lambda_{ij}(t) \Leftrightarrow \dot{\lambda}_{ij} = \lambda_{ij}(t) \{[k_j(t) + \theta k_i(t) + \rho - \eta]\} \quad (40)$$

Moreover, we can differentiate w.r.t. time the expression for the optimal R&D effort that appears in (33) to get the dynamic equation of $k_i(t)$:

$$\dot{k}_i = -\frac{1}{2\beta} \left\{ c_i(t) \dot{\lambda}_{ii} + \lambda_{ii}(t) \dot{c}_i + \theta \left[c_j(t) \dot{\lambda}_{ij} + \lambda_{ij}(t) \dot{c}_j \right] \right\}. \quad (41)$$

with $\dot{\lambda}_{ii}$ and $\dot{\lambda}_{ij}$ that obtain from (39) and (40) respectively. Moreover, from (33) we get an explicit expression for $\lambda_{ii}(t)$, i.e.

$$\lambda_{ii}(t) = -\frac{2\beta k_i(t) + \theta \lambda_{ij}(t) c_j(t)}{c_i(t)} \quad (42)$$

which can be used to further simplify (41). As to the second co-state variable, its dynamic equation (40) can be treated autonomously and, by imposing stationarity, i.e. $\dot{\lambda}_{ij} = 0$, we obtain $\lambda_{ij}(t) = 0$. This yields:

$$\dot{k}_i = \rho k_i(t) - \frac{c_i(t) q_i(t)}{2\beta} \quad (43)$$

Again, for the sake of simplicity, we introduce the symmetry assumption:

Assumption $q_i(t) = q_j(t) = q(t)$ and $k_i(t) = k_j(t) = k(t)$.

We further assume $c_i(t) = c_j(t) = c(t)$ and from (32) obtain the equilibrium per firm output $q = \frac{a - c(t)}{2 + s}$. It is now immediate to rewrite (43) as

$$\dot{k} = \rho k(t) - \frac{c(t) [a - c(t)]}{2\beta (2 + s)}. \quad (44)$$

Steady state solutions are obtained by solving the system $\dot{k} = 0$ and $\dot{c} = 0$. We start by imposing the stationarity condition $\dot{k} = 0$ to determine an equilibrium relation between $k(t)$ and $c(t)$:

$$k^{Ps}(c) = \frac{c(t) [a - c(t)]}{2\beta \rho (2 + s)} \quad (45)$$

where superscript P_s stands for *process innovation*. Hence, by substituting (45) in (6) and solve for c , we obtain three solutions, namely:

$$c = 0; \quad c = \frac{1}{2} \left\{ a \pm \frac{\sqrt{(1+\theta)[a^2(1+\theta) - 8\beta\eta\rho(2+s)]}}{1+\theta} \right\} \quad (46)$$

Before proceeding, it is immediate to note that:

Lemma 3 *The solutions $c = \frac{1}{2} \left\{ a \pm \frac{\sqrt{(1+\theta)[a^2(1+\theta) - 8\beta\eta\rho(2+s)]}}{1+\theta} \right\}$ are real if and only if $a \geq \max \left\{ c, \frac{2\sqrt{2\beta\eta\rho(1+\theta)(2+s)}}{1+\theta} \right\}$, or $\beta \leq \frac{a^2(1+\theta)}{8\eta\rho(2+s)}$.*

This Lemma is substantively equivalent to Lemma 1. It establishes that in order for the firms to reach an economically meaningful solution in steady state, the market must be sufficiently profitable or, equivalently, the marginal cost of process R&D must be low enough. Otherwise, $k^{P_s} < \eta/(1+\theta)$ and consequently $\dot{c} > 0$ forever.

We now check the stability conditions of the system. The following can be shown to hold:

Proposition 2 *Provided that $a \geq \max \left\{ c, \frac{2\sqrt{2\beta\eta\rho(1+\theta)(2+s)}}{1+\theta} \right\}$, the steady state point*

$$\begin{aligned} c^{P_s} &= \frac{1}{2} \left\{ a - \frac{\sqrt{(1+\theta)[a^2(1+\theta) - 8\beta\eta\rho(2+s)]}}{1+\theta} \right\}, \\ k^{P_s} &= \frac{\eta}{1+\theta} \end{aligned}$$

is the unique saddle point equilibrium of the game when firms invest only in process innovation.

Proof. Under symmetry, the Jacobian matrix of the dynamic system formed by (6) and (44) is:

$$J^{P_s} = \begin{bmatrix} \frac{\partial \dot{c}}{\partial c} = \eta - k(1+\theta) & \frac{\partial \dot{c}}{\partial k} = -c(1+\theta) \\ \frac{\partial \dot{k}}{\partial c} = -\frac{a-2c}{2\beta(2+s)} & \frac{\partial \dot{k}}{\partial k} = \rho \end{bmatrix} \quad (47)$$

whose trace and determinant are:

$$\begin{aligned} T(J^{Ps}) &= \rho + \eta - k(1 + \theta); \\ \Delta(J^{Ps}) &= [\eta - k(1 + \theta)]\rho - \frac{c(a - 2c)(1 + \theta)}{2\beta(2 + s)}. \end{aligned} \quad (48)$$

Evaluating $\Delta(J^{Ps})$ in (c^{Ps}, k^{Ps}) , we verify that:

$$\Delta(J^{Ps}) \propto -8\beta\eta\rho(2 + s)(1 + \theta) \quad (49)$$

where the expression on the r.h.s. is negative. Accordingly, the determinant is negative in correspondence of the steady state point (c^{Ps}, k^{Ps}) , which qualifies as the unique saddle point equilibrium of the game. ■

As in the product innovation case, we may resort to the phase diagram to illustrate saddle point stability. The locus $\dot{c} = 0$ is given by the $c = 0$ and $k = \eta/(1 + \theta)$, while the locus $\dot{k} = 0$ draws a curve in the space $\{c, k\}$ as described in Figure 2.

As in the previous setting, we now proceed to some comparative statics exercises on $k^{Ps}(c)$ and c^{Ps} , to obtain the following:

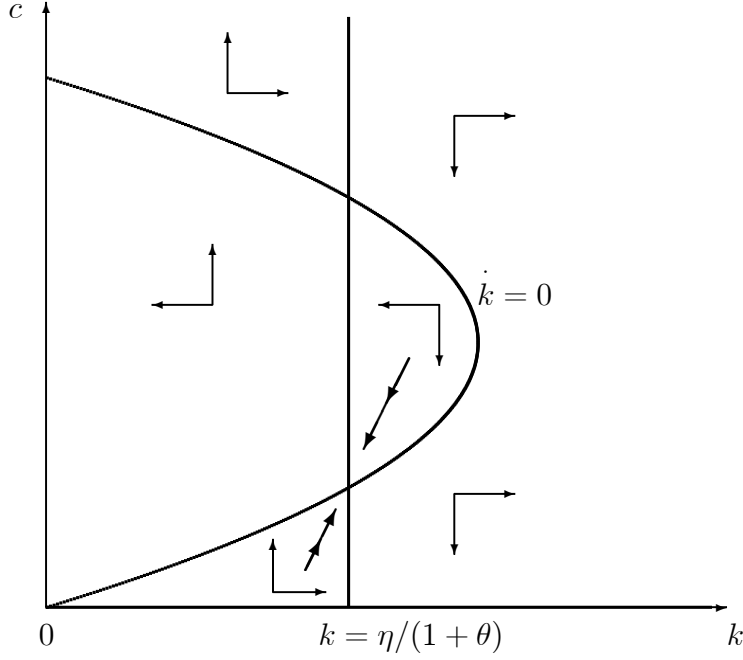
$$\frac{\partial k^{Ps}(c)}{\partial c} \propto a - 2c(t); \quad \frac{\partial k^{Ps}(c)}{\partial s} < 0; \quad \frac{\partial c^{Ps}}{\partial s} > 0. \quad (50)$$

The first derivative in (50) shows that the optimal R&D effort for process innovation is non-monotone in the level of the marginal cost, with R&D efforts initially increasing as $c(t)$ departs from c_0 (provided $c_0 \in (a/2, a)$), and then decreasing as soon as $c(t)$ goes below $a/2$.¹¹ The remaining two properties in (50) are intuitive, since (i) the higher is the substitutability between products, the lower are gross profits at any instant; as a consequence, internal funds for process R&D shrink; and (ii), steady state marginal cost is positively related to substitutability, which replicates the property we have already highlighted in the previous subsection concerning product innovation. As for (i), also

¹¹Note that c^{Ps} is surely lower than $a/2$.

in this case a higher level of substitutability decreases the available funds for financing process innovation R&D.

Figure 2 : The phase diagram in the process innovation case



3.3 Process and Product Innovation

We now analyze the most interesting case, where firms implement simultaneously process and product innovation. Instantaneous profits are given by (8) and firms maximize the discounted profit flow w.r.t. controls $x_i(t)$, $k_i(t)$ and $q_i(t)$, under the constraints represented by (4) and (6). The corresponding Hamiltonian function writes:

$$\begin{aligned}
 \mathcal{H}_i(t) = & e^{-\rho t} \{ [a - c_i(t)] q_i(t) - q_i(t)^2 - s(t) q_i(t) q_j(t) + \\
 & -\gamma [x_i(t)]^2 - \beta [k_i(t)]^2 + \lambda_i(t) s(t) [\delta - x_i(t) - x_j(t)] + \\
 & + \lambda_{ii}(t) c_i(t) [\eta - k_i(t) - \theta k_j(t)] + \lambda_{ij}(t) c_j(t) [\eta - k_j(t) - \theta k_i(t)] \}.
 \end{aligned} \tag{51}$$

Firms play simultaneously. Firm i 's first order conditions (FOCs) on controls are equivalent to (13), (14) and (33). The co-state equations for the closed-loop Nash equilibrium also coincide with those derived when treating the two types of innovation separately, i.e., (15), (39) and (40). As a consequence, the dynamic equations of controls coincide with (20) and (44), and therefore the resulting optimal controls coincide with (24) and (45), respectively, that can be used to rewrite (4) and (6) as follows:

$$\dot{s} = s \left[\delta + \rho - \frac{\sqrt{\gamma [(a-c)^2 (2-s)s + \gamma \rho^2 (2+s)^2]}}{\gamma (2+s)} \right] \quad (52)$$

$$\dot{c} = c \left[\eta - \frac{c(a-c)(1+\theta)}{2\beta\rho(2+s)} \right] \quad (53)$$

Imposing stationarity, we find the following equilibrium expressions:

$$s^*(c) = \frac{(a-c)^2 - 2\delta\gamma(\delta+2\rho) - (a-c)\sqrt{(a-c)^2 - 8\delta\gamma(\delta+2\rho)}}{(a-c)^2 + \delta\gamma(\delta+2\rho)} \quad (54)$$

$$c^*(s) = \frac{1}{2} \left\{ a - \frac{\sqrt{(1+\theta)[a^2(1+\theta) - 8\beta\eta\rho(2+s)]}}{1+\theta} \right\} \quad (55)$$

Provided that $s^*(c)$ and $c^*(s)$ belong to \mathbb{R} , then also $s^*(c) \in [0, 1]$ and $c^*(s) \in [0, a]$.

However, note that

$$s^*(c) \in \mathbb{R} \text{ iff } a \geq a_s \equiv c + 2\sqrt{2\delta\gamma(\delta+2\rho)} \quad (56)$$

$$c^*(s) \in \mathbb{R} \text{ iff } a \geq \max \left\{ c, a_c \equiv \frac{2\sqrt{2\beta\eta\rho(1+\theta)(2+s)}}{1+\theta} \right\} \quad (57)$$

The above conditions replicate those considered in Lemmas 1 and 3, respectively. Given that the two conditions are simultaneously present in this setup, in order for the R&D portfolio of the firm to generate a steady state solution it must be that $a > \max\{a_c, a_s\}$. Otherwise, at least one of the state equations (or even both) will remain positive forever. This is laid out in the following Lemma:

Lemma 4 *Firms' incentives towards R&D for process and product innovation can be characterized in terms of market size:*

- *if $a > \max \{a_s, \max \{c, a_c\}\}$, the dynamic system reaches a steady state identifying an internal solution where $c^* \in (0, a)$ and $s^* \in (0, 1)$;*
- *if $a_s > \max \{c, a_c\}$ and $a \in (\max \{c, a_c\}, a_s)$, process R&D converges to $c^* \in (0, a)$ while \dot{s} remains positive forever;*
- *if $a_s < \max \{c, a_c\}$ and $a \in (a_s, \max \{c, a_c\})$, product R&D converges to $s^* \in (0, 1)$ while \dot{c} remains positive forever;*
- *if $a < \min \{a_s, \max \{c, a_c\}\}$, both state equations remain positive forever.*

To understand the qualitative message conveyed by the above Lemma, it is worth observing that $a_s > c$ always - and therefore the constraint on the reservation price that allows the firms to reach an admissible steady state degree of product differentiation will be binding - while a_c may or may not be larger than c . If it is not, then any $a > c$ will drive an R&D process leading to an admissible steady state solution along the marginal cost dimension. In the light of these considerations, Lemma 4 states that, in general, the incentive for cost-reducing investment is relatively higher than the incentive to increase product differentiation. The economic intuition for this result can be found in the fact that, while investing in product differentiation entails a full spillover via the demand function, any effort aimed at decreasing marginal cost involves only a partial spillover measured by $\theta \in [0, 1]$.

Now we focus on the situation where real solutions exist on both sides, to evaluate the bearings of initial conditions on the optimal investment portfolio of firms. That is, provided both types of innovations are viable, shall we expect to observe either one or the other, or both in equilibrium?

To address this question, we may carry out a partition of the state space $\{c, s\}$ taking into account the *a priori* admissible domains for states, whereby $c \in [0, a]$ and $s \in [0, 1]$, so as to study the equilibrium solutions $(c^*(s), s^*(c))$, as is done in Figure 3. Both (54) and (55) are concave in the other state variable. The locus (55) is parabolic, and can be inverted to look as follows:

$$s(c^*) = \frac{c(a-c)(1+\theta) - 4\beta\eta\rho}{2\beta\eta\rho} \quad (58)$$

which takes its maximum in correspondence of $c = a/2$ and intersects the horizontal axis at

$$c_{\mp} = \frac{a(1+\theta) \mp \sqrt{(1+\theta)[a^2(1+\theta) - 16\beta\eta\rho]}}{2(1+\theta)} \quad (59)$$

with $0 < c_- < a/2 < c_+ < a$ provided that condition (57) holds.¹² Under the same condition, the maximum of $s(c^*)$ is higher than one, since

$$s(c^*) - 1|_{c=\frac{a}{2}} = \frac{a^2(1+\theta) - 24\beta\eta\rho}{8b\eta\rho} \quad (60)$$

which is surely positive as it is equivalent to (57) in $s = 1$.

The curve (54) is monotonically increasing and convex in c , as it can be ascertained by examining its first and second derivatives.¹³ In $c = 0$, (54) is

$$s^*(c) = \frac{a^2 - 2\delta\gamma(\delta + 2\rho) - a\sqrt{a^2 - 8\delta\gamma(\delta + 2\rho)}}{a^2 + \delta\gamma(\delta + 2\rho)} \propto 4\delta\gamma[a^2 + \delta\gamma(\delta + 2\rho)](\delta + 2\rho) > 0 \quad (61)$$

This and

$$1 - s^*(c) = \frac{3\delta\gamma(\delta + 2\rho) + (a-c)\sqrt{(a-c)^2 - 8\delta\gamma(\delta + 2\rho)}}{(a-c)^2 + \delta\gamma(\delta + 2\rho)} > 0 \quad (62)$$

jointly imply $s^*(c) \in (0, 1)$ in the parameter region satisfying the reality condition (56).

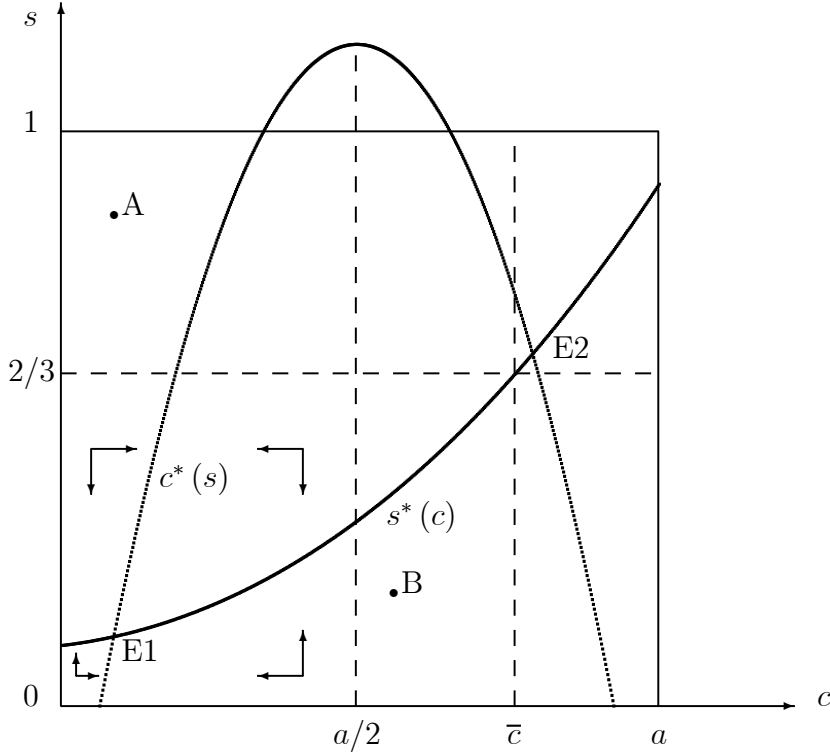
¹²This can be quickly ascertained by noting that $a_c \in \left[\frac{4\sqrt{\beta\eta\rho(1+\theta)}}{1+\theta}, \frac{2\sqrt{6\beta\eta\rho(1+\theta)}}{1+\theta} \right]$ for all $s \in [0, 1]$.

Hence, $c_{\mp} \in \mathbb{R}^+$ if a satisfies (57).

¹³These expressions are overlong, and we omit them for the sake of brevity.

Therefore, the behavior of the two loci in the rectangular region $[0, a] \times [0, 1]$ is indeed as depicted in Figure 3, where they intersect in $E1$ and $E2$.

Figure 3 : Process and Product Innovation



There remains to establish whether the intersection identified by point E_2 is admissible or not. To ascertain this, observe that

- condition (56) requires $(a - c)^2 - 8\delta\gamma(\delta + 2\rho) \geq 0$, that is, $a > c + 2\sqrt{2\delta\gamma(\delta + 2\rho)}$ or equivalently $c \in [0, a - 2\sqrt{2\delta\gamma(\delta + 2\rho)}]$; in correspondence of the upper bound of this interval, (54) yields $s^*(c) = 2/3$.
- accordingly, simplifying (45) and the sign of \dot{c} at $(c \equiv \bar{c} = a - 2\sqrt{2\delta\gamma(\delta + 2\rho)}, s = 2/3)$,

we obtain

$$k = \frac{3 \left[a\sqrt{2\delta\gamma(\delta+2\rho)} - 4\delta\gamma(\delta+2\rho) \right]}{8\beta\rho} \quad (63)$$

$$\dot{c} = \bar{c} \left[\frac{3 \left[a\sqrt{2\delta\gamma(\delta+2\rho)} - 4\delta\gamma(\delta+2\rho) \right] (1+\theta)}{8\beta\rho} + \eta \right]$$

whereby $a^2 > 8\delta\gamma(\delta+2\rho)$ ensures $k > 0$ and $\dot{c} > 0$. This, in turn, suffices to imply that the point $(c = \bar{c}, s = 2/3)$ lies below the locus $\dot{c} = 0$ and therefore point $E2$ can be dismissed.

This identifies four regions in the relevant space of states. Since $s^*(c)$ solves $\dot{s} = 0$, for all $s > s^*(c)$ we know that $\dot{s} < 0$. This implies that in such a region, firms invest in product innovation. Likewise, $c^*(s)$ solves $\dot{c} = 0$; accordingly, for all $c > c^*(s)$, we have $\dot{c} < 0$. In this region, firms invest in cost-reducing R&D activities. These two regions overlap in the cone to the north-east of the equilibrium point E_1 . Therefore, if initial conditions (c_0, s_0) identify a point belonging to the cone, then firms activate both types of investment from the very outset to the steady state. Otherwise, if, say, initial conditions identify point A to the north-west of both equilibrium loci, then firms initially invest only in product differentiation. Depending upon whether optimal trajectories cross the cone or not, firms may reach the steady state either (i) carrying out only product innovation, allowing the marginal cost to adjust via the exogenous depreciation rate, or (ii) carrying out both. The same argument goes through, *mutatis mutandis*, if initial conditions identify a point like B. In the residual region to the south-west of the equilibrium point, firms are driven to the steady state by the depreciation rates. This happens either because they do not invest, or because they do, but too little to make up for depreciation in one or both dimensions.

Concerning the issue of complementarity between product and process innovation carried out within a single firms, observe that the Hamiltonian (51) is additively separable in R&D efforts in the two directions, $\{x_i(t), k_i(t)\}$. Therefore, from FOCs (14) and (33),

one easily verifies that

$$\frac{\partial^2 \mathcal{H}_i(t)}{\partial x_i(t) \partial k_j(t)} = \frac{\partial^2 \mathcal{H}_i(t)}{\partial k_i(t) \partial x_j(t)} = 0, \quad (64)$$

which would seemingly imply that the two R&D activities are orthogonal to each other. This, however, is true only concerning the instantaneous necessary conditions for optimality. If one looks instead at the solutions of dynamic optimality conditions given by the differential equations of controls (20) and (44), i.e., $x^*(s)$ and $k^*(c)$ in (29) and (50), one finds:

$$\frac{\partial x^*(s)}{\partial c} < 0; \frac{\partial k^*(c)}{\partial s} < 0, \quad (65)$$

which also allows to establish:

$$\left| \frac{\partial k^*(c)}{\partial s} \right| > \left| \frac{\partial x^*(s)}{\partial c} \right|$$

for all $\beta \in \left(0, \frac{c\sqrt{\gamma [\gamma\rho^2 (2+s)^2 + (a-c)^2 (2-s)s]}}{\rho(2-s^2)s} \right)$. (66)

The above properties immediately entail:

Proposition 3 *Process and product R&D activities are complementary in the neighborhood of the steady state equilibrium. If the marginal cost of process innovation is low enough, then the cross effect of a change in product substitutability on process R&D is larger than the cross effect of a change in marginal cost on product R&D (in absolute value), and conversely.*

This result is qualitatively close to previous research carried out in static models of product and process innovation (Lin and Saggi 2002; Lambertini 2003, *inter alia*), although, in the present case, complementarity between the two activities is the outcome of dynamic optimization rather than the backward induction method typically employed in multistage static games.

Finally, we come to the evaluation of the stability properties of the complete model with both types of innovation. Given that, in this case, the relevant Jacobian matrix is 4×4 , one cannot draw the phase diagram. The Jacobian matrix is:

$$J^* = \begin{bmatrix} \frac{\partial \dot{s}}{\partial s} = \delta - 2x & \frac{\partial \dot{s}}{\partial x} = -2s & \frac{\partial \dot{s}}{\partial c} = 0 & \frac{\partial \dot{s}}{\partial k} = 0 \\ \frac{\partial \dot{x}}{\partial s} = \frac{(a-c)^2(3s-2)}{2\gamma(2+s)^3} & \frac{\partial \dot{x}}{\partial x} = \rho + 2x & \frac{\partial \dot{x}}{\partial c} = \frac{(a-c)(2-s)s}{2\gamma(2+s)^2} & \frac{\partial \dot{x}}{\partial k} = 0 \\ \frac{\partial \dot{c}}{\partial s} = 0 & \frac{\partial \dot{c}}{\partial x} = 0 & \frac{\partial \dot{c}}{\partial c} = \eta - k(1+\theta) & \frac{\partial \dot{c}}{\partial k} = -c(1+\theta) \\ \frac{\partial \dot{k}}{\partial s} = \frac{c(a-c)}{2\beta(2+s)^2} & \frac{\partial \dot{k}}{\partial x} = 0 & \frac{\partial \dot{k}}{\partial c} = -\frac{a-2c}{2\beta(2+s)} & \frac{\partial \dot{k}}{\partial k} = \rho \end{bmatrix} \quad (67)$$

whose characteristic equations yields four eigenvalues. Unfortunately, assessing the sign of such eigenvalues analytically is not feasible as their expressions are cumbersome. Moreover, we cannot obtain the explicit solutions for c^* and s^* as a function of parameters only, as the system

$$\begin{aligned} c - c^*(s) &= 0 \\ s - s^*(c) &= 0 \end{aligned} \quad (68)$$

is made up by equations whose degree is higher than four.

However, we may resort to numerical calculations, which can be performed as follows.

We use the solutions:

$$x^* = \frac{\delta}{2}; k^* = \frac{\delta}{1+\theta} \quad (69)$$

and set the numerical values of parameters:

$$a = 1; \beta = \gamma = \frac{1}{2}; \theta = \frac{1}{5}; \delta = \rho = \frac{1}{20}. \quad (70)$$

Then, we may (i) solve the system (68) numerically, and (ii) compute the eigenvalues $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ of J^* . Given (70), we have:

$$c^* \simeq 0.0042; s^* \simeq 0.0076 \quad (71)$$

$$\zeta_1 \simeq 0.1288; \zeta_2 \simeq 0.0808 \quad (72)$$

$$\zeta_3 \simeq -0.0308; \zeta_4 \simeq -0.0288.$$

Using instead:

$$a = 8; \beta = \gamma = \frac{1}{2}; \theta = \frac{1}{2}; \delta = \frac{1}{20}; \rho = \frac{1}{40}, \quad (73)$$

we obtain:

$$c^* \simeq 0.002; s^* \simeq 0.0001 \quad (74)$$

$$\zeta_1 \simeq 0.0999; \zeta_2 \simeq 0.0499 \quad (75)$$

$$\zeta_3 \simeq -0.0249; \zeta_4 \simeq -0.0249.$$

In general, repeating the same exercise for admissible parameter values, one can verify that the outcome is regularly $\zeta_1, \zeta_2 > 0$ while $\zeta_3, \zeta_4 < 0$. Therefore, the equilibrium is a saddle point.

4 The Bertrand Game

Under price competition, the relevant demand function for firm i is (3). The corresponding Hamiltonian function is:

$$\begin{aligned} \mathcal{H}_i(t) = e^{-\rho t} & \left\{ [p_i(t) - c(t)] \left[\frac{a}{1+s(t)} - \frac{p_i(t)}{1-[s(t)]^2} + \frac{s(t)p_j(t)}{1-[s(t)]^2} \right] + \right. \\ & -\gamma [x_i(t)]^2 - \beta [k_i(t)]^2 + \lambda_i(t)s(t) [\delta - x_i(t) - x_i(t)] + \\ & \left. + \lambda_{ii}(t)c_i(t) [\eta - k_i(t) - \theta k_j(t)] + \lambda_{ij}(t)c_j(t) [\eta - k_j(t) - \theta k_i(t)] \right\}. \end{aligned} \quad (76)$$

For brevity, we investigate only the case where product and process innovation are carried out together. Also here, feedbacks exclusively appear in the co-state equation pertaining

to the shadow prices attached to product innovation.¹⁴ The FOCs are:

$$\begin{aligned}\frac{\partial \mathcal{H}_i(t)}{\partial p_i(t)} &= \frac{a[1-s(t)] + c_i(t) - 2p_i(t) + s(t)p_j(t)}{1-[s(t)]^2} = 0 \\ \Rightarrow p_i^*(t) &= \frac{a[1-s(t)] + c_i(t) + s(t)p_j(t)}{2}\end{aligned}\quad (77)$$

$$\frac{\partial \mathcal{H}_i(t)}{\partial x_i(t)} = -2\gamma x_i(t) - \lambda_i(t)s(t) = 0 \Rightarrow x_i^*(t) = -\frac{s(t)\lambda_i(t)}{2\gamma} \quad (78)$$

$$\frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} = -2\beta k_i(t) - \lambda_{ii}(t)c_i(t) - \lambda_{ij}(t)\theta c_j(t) = 0 \Rightarrow k_i^*(t) = -\frac{[c_i(t)\lambda_{ii}(t) + \theta\lambda_{ij}(t)c_j(t)]}{2\beta} \quad (79)$$

$$\begin{aligned}-\frac{\partial \mathcal{H}_i(t)}{\partial s(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial p_j(t)} \frac{\partial p_j^*(t)}{\partial s(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial x_j(t)} \frac{\partial x_j^*(t)}{\partial s(t)} &= \dot{\lambda}_i - \rho\lambda_i(t) \Rightarrow \\ \dot{\lambda}_i &= \lambda_i(t)[- \delta + x_i(t) + x_j(t) + \rho] - \frac{\lambda_i(t)\lambda_j(t)s(t)}{2\gamma} +\end{aligned}\quad (80)$$

$$\begin{aligned}\left[\frac{p_i(t) - c_i(t)}{1 - (s(t))^2} \right] \left\{ \frac{2[a(1-s(t))^2 + 2s(t)p_i(t) - (1-s(t))^2]p_j(t)}{2[1-(s(t))^2]} + \right. \\ \left. \frac{s(t)[a-p_i(t)][1-(s(t))^2]}{2[1-(s(t))^2]} \right\} \\ - \frac{\partial \mathcal{H}_i(t)}{\partial c_i(t)} = \dot{\lambda}_{ii} - \rho\lambda_{ii}(t) \Leftrightarrow\end{aligned}\quad (81)$$

$$\begin{aligned}\dot{\lambda}_{ii} &= -\frac{p_i(t) - s(t)p_j(t) - a[1-s(t)]}{1-[s(t)]^2} + \lambda_{ii}(t)[k_i(t) + \theta k_j(t) + \rho - \eta] \\ -\frac{\partial \mathcal{H}_i(t)}{\partial c_j(t)} &= \dot{\lambda}_{ij} - \rho\lambda_{ij}(t) \Leftrightarrow \dot{\lambda}_{ij} = \lambda_{ij}(t)\{[k_j(t) + \theta k_i(t) + \rho - \eta]\},\end{aligned}\quad (82)$$

with (82) establishing that $\lambda_{ij}(t) = 0$ at all times. Transversality conditions are the same as in the Cournot game.

Proceeding in the same way as in the previous section, we can derive the dynamics of controls, which look as follows:¹⁵

$$\dot{x} = x(\rho + x) - \frac{s(t)(a-c)^2[2-s(t)(1-s(t))]}{4\gamma[2-s(t)]^2[1+s(t)]^2} \quad (83)$$

¹⁴We omit the proof of this feature, which is analogous to that of Lemma 2.

¹⁵The two dynamics equations of R&D investments are written under symmetry of R&D controls and marginal costs.

$$\dot{k} = \rho k(t) - \frac{c(t) [a - c(t)]}{2\beta [2 - s(t)] [1 + s(t)]}. \quad (84)$$

Imposing stationarity on the above differential equations, we obtain the equilibrium controls expressed in terms of the states (we omit the indication of time for brevity):

$$x_B^*(s) = \frac{-\gamma\rho [2 + s(1 - s)] + \sqrt{\gamma [2 + s(1 - s)] \{ \gamma\rho^2 [2 + s(1 - s)] + (a - c)^2 s \}}}{2\gamma [2 + s(1 - s)]} \quad (85)$$

$$k_B^*(c) = \frac{c(a - c)}{2\beta\rho (2 - s) (1 + s)} \quad (86)$$

These expressions can be fruitfully compared to the corresponding ones in the Cournot game, i.e. (24) and (45), to verify that

$$\begin{aligned} x_B^*(s) - x_C^*(s) \propto & 8\gamma\rho^2(2 - s)^2(1 + s)^2(2 + s)^2 [8 + s(1 + s)(4 + s(10 - s(3 - s)))] + \\ & + (a - c)^2 s [24 + s(12 + s(18 + s(13 + s^2)))]^2 \end{aligned} \quad (87)$$

which is positive for all admissible levels of a , c and s . Likewise:

$$k_B^*(s) - k_C^*(s) \propto \frac{c(a - c) s^2}{2\beta\rho (2 - s) (2 + s) (1 + s)} > 0. \quad (88)$$

This discussion entails the following:

Proposition 4 *For any given levels of c and s , Bertrand competition provides firms with higher incentives to invest both in product and in process innovation than Cournot competition.*

The intuitive explanation of the above result is that price competition is harsher than quantity competition, thus calling for higher R&D efforts to reduce both marginal costs and product substitutability in order to safeguard profits. This, in terms of the existing literature on this issue, has a definite Arrovian flavour in that intensifying market competition enhances R&D.

5 Concluding Remarks

We have investigated the product/technology life cycle by using a differential game in which firms may invest in product and process innovation. We have considered horizontal product innovation that reduces product substitutability, and process innovation reducing marginal production cost. The solution concept we have adopted is the closed-loop Nash equilibrium.

The main question we have addressed, in the light of the existing debate based upon empirical evidence, is about the relative timing of adoption of product and process innovations. We have shown that firms may activate both activities simultaneously, or one at a time, depending upon the initial conditions of the state variables, i.e., initial levels of product homogeneity *vis à vis* the marginal cost of production. In particular, four situations may arise: (i) firms activate both types of investment from the very outset to the steady state; (ii) firms initially invest only in product differentiation and then reach the steady state either carrying out only product innovation or carrying out both; (iii) firms initially invest only to reduce the cost of production and then arrive at the steady state either carrying out only process innovation or carrying out both; (iv) firms do not invest at all in either type of innovation, and they are driven to the steady state by the depreciation rate. The taxonomy of paths towards the equilibrium generated by the theoretical setup we have analyzed is in line with (and provides further theoretical support to) the conclusions drawn by Adner and Levinthal (2001), pointing out that product/technology life cycle may not necessarily follow the first-product-then-process pattern outlined by conventional wisdom.

The comparison of R&D incentives under Cournot and Bertrand competition has clearly shown that softening market competition entails lower R&D efforts in both directions, whereby the present analysis supports the Arrovian view.

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