Oligopoly with Hyperbolic Demand: A Differential Game Approach[1]

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Abstract

Convex demand functions, although commonly used in consumer theory and in accordance with a large amount of empirical evidence, are known to be problematic in the analysis of firms' behaviour and therefore rarely used in oligopoly theory, due to the possible lack of concavity of firms' profit functions and the indeterminacy arising in the limit as marginal costs tend to zero. Here I investigate a dynamic oligopoly model with hyperbolic demand and sticky price, characterising the open-loop optimal control and the related steady state equilibrium, to show that the indeterminacy associated with the limit of static model is indeed confined to the steady state of the dynamic one, while the latter allows for a well-behaved solution at any time during the game. Although the feedback solution cannot be analytically attained since the model is not built in linear-quadratic form, I show that analogous considerations also apply to the Bellman equation of the individual firm.

JEL Codes: C73, D43, D92, L13

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1 Introduction

Most of the existing literature on oligopoly theory (either static or dynamic) assumes linear demand functions, as this, in addition to simplifying calculations, also ensures both concavity and unicity of the equilibrium, which, in general, wouldn't be warranted in presence of convex demand systems (see Friedman, 1977; and Dixit, 1986, *inter alia*). However, the use of linear demand function is in sharp contrast with the standard microeconomic approach to consumer behavior, where the widespread adoption of Cobb-Douglas preferences (or their log-linear affine transformation) yields hyperbolic demand functions. The same applies to the so-called quasi-linear utility function, concave in consumption and linear in money, that again yields a convex demand system. Indeed, both preference structures share the common property of producing isoelastic demand functions.[1] In fact, this is sometimes openly referred to in the field of industrial organization, where researchers mentions the opportunity of dealing with non-linear demand functions, and then promptly leave it aside for the sake of tractability.[2] Additionally, the econometric approach to demand theory has produced the highest efforts to building up a robust approach to the estimation of non-linear individual and market demand functions, yielding a large empirical evidence in this direction.[3] With these considerations in mind, it appears desirable to investigate the bearings of non-linear demand systems on the performance of firms operating in oligopolistic markets, using thus a setup with solid microfoundations corroborated by robust empirical evidence, even though this is a costly approach in terms of analytical tractability.

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[1] For a thourough illustration of these issues in consumer theory, see the classical textbooks of Deaton and Muellbauer (1980), and Varian (1992), *inter alia*.

[2] A noteworthy example in this respect is Shy (1995, pp. 53-54), using quasi-linear utility function to define the concept of consumer surplus.

[3] See Hausman (1981) and Varian (1982, 1990), *inter alia*.

With specific reference to differential games, the use of linear demand functions (jointly with either linear or quadratic cost functions) allows for the closed-form solution of the feedback equilibrium through the Bellman equation of the representative firm, as the model takes a linear-quadratic form and therefore one can stipulate that the corresponding candidate value function is also linear-quadratic. However, there is no particular reason to believe that a linear function describes correctly virtually any market demand in the real world, and therefore it is of primary interest to design, if possible, closed-form solutions of market games with non-linear demand functions.[4]

The aim of this paper is to illustrate a way out of the aforementioned problem, offered by dynamic game theory. To this purpose, I propose a differential oligopoly game where firms face implicit menu costs of adjusting outputs over time because market price is sticky (as in Simaan and Takayama, 1978; and Fershtman and Kamien, 1987, *inter alia*). The main results can be summarised as follows. Using the open-loop solution method, I prove that, for any given level of price stickiness, the first order condition on the individual firm's control yields a unique and well defined solution at any time during the game, as long as the co-state variable associated with the price dynamics differs from zero (which is necessary for Pontryagin's Maximum Principle to hold). Then, I show that

- the steady state is stable for any degree of price stickiness, provided that the number of firms in the market is at least equal to two;
- the monopoly equilibrium is stable, provided that the degree of price stickiness is below a critical threshold ensuring concavity; and

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[4] To the best of my knowledge, the only existing analysis of a differential oligopoly game with non-linear market demand is in Cellini and Lambertini (2007). There, however, production costs are taken to be nil for the sake of analytical tractability.

- indeterminacy arises at the steady state only, if the marginal production cost tends to zero.

The feedback equilibrium cannot be characterised analytically, as the game is not a linear quadratic one. However, the first order condition taken on the Bellman equation of the generic firm shows that a unique solution always exists at any time during the game, also in the limit case where marginal cost is equal to zero.

In summary, the essence of the ensuing analysis is that the static game picks a very specific and relatively uninteresting feature of firms' behaviour in presence of convex demand functions. Put it differently, the static version of the problem is unable to describe the optimal output decisions of firms *along the path* to a steady state that, though problematic as it may be, will nonetheless be reached only *asymptotically* on doomsday. Accordingly, one may view the present setup as an instance of the value added that the differential game approach may offer in terms of our understanding of this and other economic issues that lend themselves to an explicitly dynamic modelization.

The remainder of the paper is structured as follows. Section 2 summarises the static game. The open-loop solution of the differential game with sticky market price is investigated in section 3. The feedback problem is briefly accounted for in section 4. Section 5 contains some concluding remarks.

2 A summary of the static game

Consider a market where N single-product firms supply individual quantities q_i, $i = 1, 2, 3, ...N$. The good is homogeneous, and market demand is $p = a/Q$, $Q = \sum_{i=1}^{N} q_i$. This demand function is the outcome of the constrained maximum problem of a representative consumer endowed with a log-linear utilility function

$$U = Log\,[Q] + m \tag{1}$$

where m is a numeraire good whose price is normalised to one. The budget constraint establishes that the consumer's nominal income Y must be large enough to cover the expenditure, so that $Y \geq pQ + m$. The representative consumer must

$$\max_{Q} L = U + \mu\,(Y - pQ - m)\,. \tag{2}$$

Solving the above problem, one obtains indeed the hyperbolic demand function $p = a/Q$.

On the supply side, production entails a total cost cq_i^2, where $c > 0$ is a constant parameter measuring marginal production cost. Market competition takes place *à la* Cournot-Nash; therefore, firm i chooses q_i so as to maximise profits $\pi_i = (p - cq_i)\,q_i$. This entails that the following first order condition must be satisfied (assuming interior solutions):

$$\frac{\partial \pi_i}{\partial q_i} = \frac{aQ_{-i}}{(q_i + Q_{-i})^2} - 2cq_i = 0 \tag{3}$$

where $Q_{-i} \equiv \sum_{j \neq i} q_j$. The associated second order condition:

$$\frac{\partial^2 \pi_i}{\partial q_i^2} = -\frac{2a \sum_{j\neq} q_j}{(q_i + Q_{-i})^3} - 2c \leq 0 \tag{4}$$

is always met. Imposing the symmetry condition $q_i = q$ for all $q_i = 1, 2, 3, ...N$, one obtains the Cournot-Nash equilibrium

$$q^{CN} = \frac{1}{N}\sqrt{\frac{a\,(N-1)}{2c}}\,;\, p^{CN} = \frac{\sqrt{2ac\,(N-1)}}{N-1} \tag{5}$$

yielding profits $\pi^{CN} = a\,(N+1)\,/\,(2N^2)$. Note that one must exclude the case $N = 1$ as the above solution is indeterminate under monopoly. If the N firms were operating under perfect competition, then price would be $p^* = 2cq_i$ and therefore $q^* = \sqrt{a/\,(2cN)}$.

Moreover, it is apparent that the above solutions (i.e., both the Cournot-Nash equilibrium and the perfectly competitive equilibrium) are determinate for all $c > 0$, while they would become indeterminate if parameter c were nil. In the remainder, I investigate a differential game with sticky price revealing that the indeterminacy associated with the limit of the static equilibrium as c tends to zero is indeed confined to the steady state, while the Cournot-Nash output is well defined and economically meaningful at any time during the game.

3 The dynamic setup

Consider the following dynamic version of the oligopoly game examined in the previous section. At any $t \in [0, \infty)$, each of N firms supplies quantity $q_i(t)$, $i = 1, 2, 3, ...N$, of the same homogeneous good, at an instantaneous cost $C_i(t) = cq_i^2(t)$, $c > 0$. At any instant, the notional market demand function is:

$$\widehat{p}(t) = \frac{a}{Q(t)}, \; Q(t) = \sum_{i=1}^{N} q_i(t)\,; \; a > c. \tag{6}$$

However, the current market price $p(t)$ is sticky and, as in Simaan and Takayama (1978) and Fershtman and Kamien (1987),[5] its dynamics is described by the following differential equation:

$$\dot{p}(t) = s\left[\widehat{p}(t) - p(t)\right]. \tag{7}$$

The above dynamics establishes that price $p(t)$ adjusts proportionately to the difference between the *correct* price level given by the notional demand function (6) and the current price, the instantaneous speed of adjustment

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[5] See also Fershtman and Kamien (1990), Tsutsui and Mino (1990) and Cellini and Lambertini (2004).

being measured by the constant parameter $s \in [0, \infty)$.[6] Firms face implicit menu costs generated by the price stickiness, so that they never supply the correct quantities leading to the market clearing price $p(t) = \widehat{p}(t)$ (except at the steady state, of course).

The problem of firm i is to choose output $q_i(t)$ so as to maximisize its own discounted profits:

$$\Pi_i(p(t), \mathbf{q}(t)) \triangleq \int_0^\infty [p(t) - cq_i(t)]\, q_i(t)\, e^{-\rho t} dt \tag{8}$$

s.t. the price dynamics (7) and the initial condition $p(0) = p_0$. $\mathbf{q}(t)$ is the vector of all firms' controls.

The Hamiltonian of firm i is:[7]

$$\mathcal{H}_i(p(t), \mathbf{q}(t)) = e^{-\rho t} \left\{ [p(t) - cq_i(t)]\, q_i(t) + \lambda_i(t)\, s \left[\frac{a}{q_i(t) + Q_{-i}(t)} - p(t) \right] \right\} \tag{9}$$

where $\lambda_i(t) = \mu_i(t) e^{\rho t}$, and $\mu_i(t)$ is the co-state variable that firm i associates to $p(t)$.

The necessary conditions are:[8]

$$\frac{\partial \mathcal{H}_i(\cdot)}{\partial q_i(t)} = p(t) - 2cq_i(t) - \frac{\lambda_i(t)\, as}{[q_i(t) + Q_{-i}(t)]^2} = 0 \tag{10}$$

$$-\frac{\partial \mathcal{H}_i(\cdot)}{\partial p(t)} = \dot{\lambda}_i(t) - \rho \lambda_i(t) \Leftrightarrow$$

$$\dot{\lambda}_i(t) = \lambda_i(t)(\rho + s) - q_i(t) \tag{11}$$

with the transversality condition

$$\lim_{t \to \infty} e^{-\rho t} \lambda_i(t)\, p(t) = 0. \tag{12}$$

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[6] Note that, if s is infinitely high, then the price adjusts immediately to the notional level, which is equivalent to saying that price is not sticky at all.

[7] In the ensuing analysis, I will confine my attention to interior solutions.

[8] Exponential discounting is omitted for brevity.

Now, solving (10) w.r.t. $\lambda_i(t)$, we obtain:

$$\lambda_i(t) = \frac{[p(t) - 2cq_i(t)]\,[q_i(t) + Q_{-i}(t)]^2}{as}. \tag{13}$$

Before proceeding, it is worth noting that (10) and/or (13) imply that $p(t) \neq 2cq_i(t)$ for all $s \in (0, \infty)$. Or, conversely:

Lemma 1 *Provided $s \in (0, \infty)$ and $\lambda_i(t) > 0$, the price will differ from marginal cost at any time t during the game.*

The second order (sufficiency) condition for concavity requires:

$$\frac{\partial^2 \mathcal{H}_i(\cdot)}{\partial q_i^2(t)} = \frac{2\lambda_i(t)\,as}{[q_i(t) + Q_{-i}(t)]^3} - 2c \leq 0 \tag{14}$$

or, equivalently,

$$\lambda_i(t) \leq \overline{\lambda}_i(t) \triangleq \frac{c\,[q_i(t) + Q_{-i}(t)]^3}{2as}. \tag{15}$$

We can proceed to the characterisation of the control dynamics. Differentiating (10) w.r.t. time, we have:

$$\dot{p}(t) - 2c\dot{q}_i(t) - \frac{\dot{\lambda}_i(t)\,as}{Q^2(t)} + \frac{2\lambda_i(t)\,as\left[\dot{q}_i(t) + \sum_{j\neq i}\dot{q}_j(t)\right]}{Q(t)^3} = 0, \tag{16}$$

that can be solved to obtain the following description of the dynamics of firm i's control:[9]

$$\dot{q}_i = \frac{Q^3\dot{p} - a\left(\dot{\lambda}_i Q - 2\lambda\sum_{j\neq i}\dot{q}_j\right)s}{2\left(cQ^3 - as\lambda_i\right)}. \tag{17}$$

Imposing the symmetry conditions $\dot{q}_j = \dot{q}_i = \dot{q}$ and $q_j = q_i = q$, the dynamics of price simplifies as follows:

$$\dot{p} = s\left[\frac{a}{Nq} - p\right]. \tag{18}$$

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[9] Henceforth, the indication of time is omitted for brevity.

Then plugging (11), (13) and (18) into (17), the dynamics of the optimal control can be rewritten as follows:

$$\dot{q} = \frac{N^2 q \left[p\left(2s+\rho\right) - 2cq\left(s+\rho\right)\right] - as\left(N+1\right)}{2N^2\left(p-3cq\right)}. \tag{19}$$

Imposing the stationarity condition $\dot{p} = 0$, one obtains $p = a/\left(Nq\right)$, which can be plugged into (19). At the steady state, $\dot{q} = 0$ yields:

$$q_A^{ss} = 0;\ q_B^{ss} = \frac{1}{N}\sqrt{\frac{a\left[N\left(s+\rho\right)-s\right]}{2\left(s+\rho\right)c}} > 0;\ q_C^{ss} = -\frac{1}{N}\sqrt{\frac{a\left[N\left(s+\rho\right)-s\right]}{2\left(s+\rho\right)c}} < 0. \tag{20}$$

The solutions q_A^{ss} and q_C^{ss} can be disregarded for obvious reasons.[10] In the remainder, I will indicate the open-loop steady state levels of output, price and profits by the subscript OL. In correspondence of $q_{OL}^{ss} = q_B^{ss}$, the steady state price is:

$$p_{OL}^{ss} = a\sqrt{\frac{2\left(s+\rho\right)c}{a\left[N\left(s+\rho\right)-s\right]}} > 2cq_{OL}^{ss}\ \ \forall\, s \in \left(0,\infty\right). \tag{21}$$

On the basis of (20-21), we have

Lemma 2 *Provided $c > 0$, the open-loop steady state equilibrium $\left(p_{OL}^{ss}, q_{OL}^{ss}\right)$ is economically acceptable for any $N \geq 1$ and any finite value of s.*

This is a key result of the dynamic analysis carried out so far: unlike the static model, where the equilibrium outcome (5) is acceptable only for $N \geq 2$, here the steady state of the dynamic setting is also well defined for the monopoly case, under some appropriate conditions on the relevant parameters.

At $\left(p_{OL}^{ss}, q_{OL}^{ss}\right)$, steady state profits amount to:

$$\pi_{OL}^{ss} = \frac{a\left[N\left(s+\rho\right)+s\right]}{2N^2\left(s+\rho\right)}. \tag{22}$$

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[10] The first because $q = 0$ would imply an infinitely high price, the second because a negative quantity is economically meaningless.

The following result can be shown to hold:

Proposition 3 *If $N \geq 3$, the steady state identified by*

$$p_{OL}^{ss} = a\sqrt{\frac{2(s+\rho)c}{a[N(s+\rho)-s]}}\,;\, q_{OL}^{ss} = \frac{1}{N}\sqrt{\frac{a[N(s+\rho)-s]}{2(s+\rho)c}}$$

is a saddle point for all admissible values of s. If $N \in \{1,2\}$, then:

- *in $N = 1$, (p^{ss}, q^{ss}) is (i) a saddle point for all $s \in (0, \rho/2)$; (ii) an unstable focus for all $s > \rho/2$;*
- *in $N = 2$, (p^{ss}, q^{ss}) is (i) a saddle point for all $s \in (0, 2\rho)$; (ii) a stable node for all $s \in (2\rho, 2.226\rho)$; a stable focus for all $s > 2.226\rho$.*

Proof. See the Appendix. ■

What the above Proposition establishes is not only that the open-loop solution of the dynamic game is determined, but also that the associated steady state point is always stable, provided that either (i) the number of firms is sufficiently high, or (ii) the speed of price adjustment is sufficiently low. These features deserve some additional comments. Property (i) indicates that competition (i.e., $N \geq 2$) ensures stability irrespective of the level of price stickiness. This is, *per se*, an element that cannot emerge from the corresponding static game. Conversely, in the monopoly case the steady state $(p_{OL}^{ss}, q_{OL}^{ss})$ is unstable for all levels of s above a critical threshold (and conversely). The reason for this result is to be found through the analysis of the sufficiency condition (15), that can be simplified using (13), p_{OL}^{ss} and q_{OL}^{ss} to yield:

$$-\frac{2c\left[N^2\rho + s(N+1)(N-2)\right]}{N[N(s+\rho)-s]} \leq 0. \tag{23}$$

Now, imposing $N = 1$, (23) becomes:

$$-\frac{2c(\rho - 2s)}{\rho} \leq 0\,\forall s \leq \frac{\rho}{2}. \tag{24}$$

It is then straightforward to verify that (23) is always verified for $N \geq 2$.

Consequently, I may state

Proposition 4 *$(p^{ss}_{OL}, q^{ss}_{OL})$ is a maximum point for all $N \geq 2$ and all admissible values of s. If instead $N = 1$, then $(p^{ss}_{OL}, q^{ss}_{OL})$ is a maximum point for all $s < \rho/2$.*

That is, under monopoly, the problem is concave if the speed of price adjustment is low enough (i.e., in the region where $(p^{ss}_{OL}, q^{ss}_{OL})$ is a saddle point), while the concavity condition is violated elsewhere (in the region where $(p^{ss}_{OL}, q^{ss}_{OL})$ is an unstable focus). Accordingly, there emerges that instability goes intuitively along with the lack of concavity at the monopoly equilibrium.

A simple comparative statics exercise can be carried out to assess the effect of an exogenous change in s on the equilibrium levels of price, individual output and profits:

$$\begin{aligned}
\frac{\partial p^{ss}_{OL}}{\partial s} &= \frac{a^2 c\rho}{\sqrt{2c(s+\rho)\left[a(N(s+\rho)-s)\right]^3}} > 0 \\
\frac{\partial q^{ss}_{OL}}{\partial s} &= -\frac{ac\rho}{2N\sqrt{2\left[cN^2(s+\rho)\right]^3 a(N(s+\rho)-s)}} < 0 \\
\frac{\partial \pi^{ss}_{OL}}{\partial s} &= \frac{a\rho}{2N^2(s+\rho)^2} > 0
\end{aligned} \tag{25}$$

This implies:

Corollary 5 *For all $c > 0$ and $N \geq 1$, an increase in s entails a decrease in the steady state output (per firm as well as at the industry level) and a consequent increase in the steady state price. The balance of these two effects drives an increase in steady state profits.*

Of course, this also drives a decrease in consumer surplus; therefore, one can conclude that price stickiness is good for firms while being bad for consumers (and, overall, for social welfare).

Then, taking the limit of (p^{ss}, q^{ss}) as s tends to infinity, it is easily ascertained that:

$$\lim_{s\to\infty} p^{ss}_{OL} = \lim_{\rho\to 0} p^{ss}_{OL} = \frac{\sqrt{2ac\,(N-1)}}{N-1}\ ;\ \lim_{s\to\infty} q^{ss}_{OL} = \lim_{\rho\to 0} q^{ss}_{OL} = \frac{1}{N}\sqrt{\frac{a\,(N-1)}{2c}} \tag{26}$$

and

$$\lim_{s\to\infty} \pi^{ss}_{OL} = \frac{a\,(N+1)}{2N^2}\,. \tag{27}$$

Additionally:

$$\lim_{s\to 0} p^{ss}_{OL} = \lim_{\rho\to\infty} p^{ss}_{OL} = \sqrt{\frac{2ac}{N}}\ ;\ \lim_{s\to 0} q^{ss}_{OL} = \lim_{\rho\to\infty} q^{ss}_{OL} = \sqrt{\frac{a}{2Nc}}\ ; \tag{28}$$

$$\lim_{s\to 0} \pi^{ss}_{OL} = \lim_{\rho\to\infty} \pi^{ss}_{OL} = \frac{a}{2N} \tag{29}$$

corresponding to the perfectly competitive outcome of the static game.

These results prove the following additional Corollary to Proposition 3, describing the limit behaviour of the model, in correspondence of (i) instantaneous price adjustment, or (ii) no price adjustment at all, alternatively:

Corollary 6 *In the limit,*

- *as s tends to infinity (or ρ tends to zero), the steady state coincides with the static Cournot-Nash equilibrium, which is economically admissible for all $N \geq 2$;*

- *as s tends to zero (or ρ tends to infinity), the steady state coincides with the static perfectly competitive equilibrium, which is admissible for all $N \geq 1$.*

These limit results are broadly in accordance with the corresponding results obtained by Fershtman and Kamien (1987) and Cellini and Lambertini (2004) in the model with linear demand and decreasing returns to scale.

Finally, observe that q^{ss}_{OL} becomes indeed indeterminate in the limit case where c tends to zero, as it happens at the corresponding solution of the static game. This means that the steady state of the static game precisely portrays, in this special case, the outcome of the static setup.

4 The feedback solution: sketch

As stated above, the Bellman equation that would yield the feedback equilibrium of the game cannot be solved analytically since the game at hand is not a linear-quadratic one. However, a relevant implication of the first order condition can be easily drawn.

The Bellman equation for firm i is:

$$\rho V_i\left(p\left(t\right)\right) = \max_{q_i(t)} \left\{ \left[p\left(t\right) - cq_i\left(t\right)\right] q_i\left(t\right) + V_i'\left(p\left(t\right)\right) s \left[\frac{a}{q_i\left(t\right) + \sum_{j\neq} q_j\left(t\right)} - p\left(t\right)\right]\right\} \tag{30}$$

where $V_i\left(p\right)$ is the value function and $V_i'\left(p\right) = \partial V_i\left(p\left(t\right)\right)/\partial p\left(t\right)$.[11] Now, taking the first order condition, we have:

$$p\left(t\right) - 2cq_i\left(t\right) - \frac{V_i'\left(p\left(t\right)\right) as}{\left[q_i\left(t\right) + \sum_{j\neq} q_j\left(t\right)\right]^2} = 0, \tag{31}$$

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[11] To this regard, it is appropriate to stress that $V_i'\left(p\left(t\right)\right)$ and $\lambda_i\left(t\right)$ are not equivalent, as shown by Caputo (2007): while $V_i'\left(p\left(t\right)\right)$ can be considered a shadow price when looking at the feedback solution, this is not true for $\lambda_i\left(t\right)$, unless the game belongs to the class of *perfect* or *state-redundant* games producing strongly time consistent open-loop Nash equilibria. For a survey of perfect or state-redundant games, see Docker *et al.* (2000, ch. 7).

which admits a unique real solution w.r.t. $q_i(t)$, for any admissible value of the marginal cost, including $c = 0$:

$$q_i^*(t) = \frac{p(t)\left[\sqrt[3]{\varpi} - 4c\sum_{j\neq} q_j(t)\right] - p^2(t) - \left[\sqrt[3]{\varpi} + 2c\sum_{j\neq} q_j(t)\right]^2}{6c\sqrt[3]{\varpi}} \in \mathbb{R}^+ \tag{32}$$

where the expression for ϖ corresponds to the expression for φ, except that $V_i'(p(t))$ replaces $\lambda_i(t)$.

5 Concluding remarks

I have revisited the Cournot oligopoly with isoelastic demand function using a dynamic approach based upon the assumption of price stickiness. This setup, which is in line with the standard microeconomic approach to consumer theory and is supported by empirical evidence, has allowed me to show that the indeterminacy issue commonly imputed to the static version of the game does not arise in general in the present dynamic reformulation, with the exception of the instant in which firms reach the steady state. In the open-loop case, at any other time during the game, the presence of co-state variables (which must differ from zero on the basis of the Maximum Principle) in the first order conditions ensure the existence of a well defined solution for firms' optimal outputs. Likewise, an analogous conclusion obtains in the feedback case, although the analytical solution remains of course out of reach since the game is not defined in a linear-quadratic form.

Appendix

Proof of Proposition 3 In order to study the stability properties of the model, I impose symmetry on quantities, $q_i = q_j = q$, whereby I obtain a 2×2 dynamic system consisting of the state equation (18) and the control equation (19). The associated Jacobian matrix is:

$$J = \begin{bmatrix} \dfrac{\partial \dot{p}}{\partial p} & \dfrac{\partial \dot{p}}{\partial q_i} \\ \dfrac{\partial \dot{q}_i}{\partial p} & \dfrac{\partial \dot{q}_i}{\partial q_i} \end{bmatrix} \tag{a1}$$

At (p^{ss}, q^{ss}), the trace and determinant of matrix J are:

$$\begin{aligned} T(J) &= \frac{s(s-2\rho) - N(s-\rho)(s+\rho)}{N(s+\rho) - 3s} \\ \Delta(J) &= -\frac{2s(s+\rho)\left[N(s+\rho) - s\right]}{N(s+\rho) - 3s} \end{aligned} \tag{a2}$$

Now note that the numerator of $\Delta(J)$ is positive for all $N > s/(s+\rho)$, which is always true as $N \geq 1$; hence, $\Delta(J) < 0$ if $N > \max\{1, 3s/(s+\rho)\}$, with $\lim_{s\to\infty} 3s/(s+\rho) = 3$. Accordingly, $N \geq 3$ suffices to ensure that $\Delta(J) < 0$ and, as a result, the pair (p^{ss}, q^{ss}) identifies a saddle point for all $N \geq 3$.

Otherwise, if $N \in \{1, 2\}$,

$$\begin{aligned} \frac{3s}{s+\rho} &\geq 1 \,\forall s \geq \frac{\rho}{2} \\ \frac{3s}{s+\rho} &\geq 2 \,\forall s \geq 2\rho \end{aligned} \tag{a3}$$

and one has to take into account the sign of $T(J)$, $\Delta(J)$ and $[T(J)]^2 - 4\Delta(J)$ together:

- $N = 1$. In this case,

$$T(J) = \rho > 0, \tag{a4}$$

$$\Delta(J) = \frac{2s\rho(s+\rho)}{2s-\rho} \lesseqgtr 0 \;\forall s \lesseqgtr \frac{\rho}{2}, \tag{a5}$$

$$[T(J)]^2 - 4\Delta(J) \gtrless 0 \ \forall s \lessgtr \frac{\rho}{2}. \tag{a6}$$

Hence, in the monopoly case (p^{ss}, q^{ss}) is (i) a saddle point for all $s < \rho/2$, (ii) an unstable focus for all $s > \rho/2$.

- $N = 2$. In this case,

$$T(J) = \frac{2\rho(\rho - s) - s^2}{2s - \rho} > 0 \ \forall s \in \left(-\left(1+\sqrt{3}\right)\rho, \left(\sqrt{3}-1\right)\rho\right), \tag{a7}$$

$$\Delta(J) = \frac{2s(s+\rho)(s+2\rho)}{s - 2\rho} \lessgtr 0 \ \forall s \lessgtr 2\rho, \tag{a8}$$

$$[T(J)]^2 - 4\Delta(J) \gtrless 0 \ \forall s \lessgtr 2.226\rho. \tag{a9}$$

Hence, in the duopoly case (p^{ss}, q^{ss}) is (i) a saddle point for all $s < 2\rho$, (ii) a stable node for all $s \in (2\rho, 2.226\rho)$; a stable focus for all $s > 2.226\rho$.

This concludes the proof.■

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