On the Time Consistency of Equilibria in a Class of Additively Separable Differential Games*

Emanuele Bacchiega, Luca Lambertini and Arsen Palestini
Department of Economics, University of Bologna
Strada Maggiore 45, 40125 Bologna, Italy
emanuele.bacchiega@unibo.it; luca.lambertini@unibo.it; palestini@math.unifi.it

Abstract

A class of state-redundant differential games games is detected, where players can be partitioned into two groups, so that the state dynamics and the payoff functions of all players are additively separable w.r.t. controls and states of any two players belonging to different groups. We prove that, in this class of games, open-loop Nash and feedback Stackelberg equilibria coincide, both being strongly time consistent. This allows us to bypass the issue of the time inconsistency that typically affects the open-loop Stackelberg solution.

JEL Classification: C73

Keywords: differential games, time consistency, additive separability

*We thank George Leitmann, an anonymous referee, Gustav Feichtinger, Morton Kamien and the audience at SING4 (Wroclaw, June 26-28, 2008) for helpful comments and suggestions. The usual disclaimer applies.

1 Introduction

The time consistency of equilibria is a crucial and long standing issue in dynamic game theory. In general, open-loop Nash (i.e., simultaneous play) equilibria are time consistent while open-loop Stackelberg (i.e., sequential play) equilibria are time inconsistent, and there exists a relevant stream of literature investigating special classes of games where these problems do not arise. Starting from Clemhout and Wan (1974), several types of games producing strongly time consistent (or subgame perfect) Nash equilibria under open-loop information have been identified.[1] Attaining time consistency in Stackelberg games is a somewhat more challenging enterprise. After the seminal contributions of Simaan and Cruz (1973a,b), the idea that hierarchical dynamic games yield time inconsistent Stackelberg equilibria has dominated the related literature in economics and, more generally, in the social sciences as a whole.

Here we focus our attention on dynamic games in continuous time, i.e., differential games. Our aim is to characterise a class of games which are additively separable with respect to state and control variables, in the following sense. We model Stackelberg play as carried out by agents partitioned into two groups, labelled as *leaders* and *followers*, respectively, and we assume that players belonging to the same group move simultaneously while agents belonging to different groups move sequentially. In this framework, we adopt a specific definition of additive separability of the state dynamics and the instantaneous payoff functions w.r.t. controls and states. In particular, we exclude any multiplicative interplay between any variables (either states or controls) pertaining to players belonging to different groups.[2] Such definition ensures that the resulting instantaneous best response functions of any two players belonging to different groups are orthogonal to each other in the space of controls. On this basis, we prove two main results:

- If a differential game satisfies our definition of additive separability, then its feedback Nash and Stackelberg equilibria coincide.
- Additionally, if the open-loop Nash solution of the same game is strongly time consistent, then the feedback Stackelberg equilibrium collapses

[1] Classes of tractable open-loop differential games have been identified by Dockner *et al.* (1985). Several other contributions illustrate specific games whose open-loop solutions are strongly time consistent. For exhaustive surveys of such games, see Mehlmann (1988, ch. 4) and Dockner *et al.* (2000, ch 7).

[2] This definition permits the presence of multiplicative effects between any two or more variables of players within the same group.

onto the open-loop Nash one, precisely because the latter also coincides with the feedback Nash equilibrium.

Note that the above points hold irrespective of whether the open-loop Stackelberg solution is time consistent or not. Accordingly, this makes unnecessary to deal explicitly with the eventual time inconsistency that might well affect the open-loop hierarchical game, as the feedback Stackelberg equilibrium, which is strongly time consistent by definition, can be easily characterised by solving the open-loop Nash setup.

Our main result is connected with Rubio (2006), where, in the two-player case, it is shown that the orthogonality of instantaneous best response functions yields the coincidence of feedback Nash and Stackelberg equilibria. What we do here is to characterise an underlying structure whereby a game produces orthogonal best replies across groups, in the general case with n players. Additionally, a consideration which is central in Rubio's paper is that the coincidence of feedback Nash and Stackelberg equilibria happens even too often in economic problems,[3] and leaves little room for the analysis of sequential moves. Yet, our point is that this coincidence may instead be considered as an advantage as it allows us to charasterise analytically feedback Stackelberg equilibria by solving the feedback Nash equilibria of the same games. Furthermore, if the Nash open-loop solution is strongly time consistent, then the orthogonality of instantaneous best response functions yields that the open-loop Nash equilibrium coincides with the feedback Stackelberg equilibrium which is strongly time consistent as well.

2 The basic setup

Consider an infinite horizon differential game with the following features:

- n is the number of players;
- $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t)) \in X \subset \mathbb{R}^n$, where X is an open set, is a vector of state variables;
- $\mathbf{u}(t) \in U := U_1 \times \ldots \times U_n$, where U_i is an open set for every $i = 1, \ldots, n$, is a vector of control variables; $u_i(t)$ is the i-th player's control;

[3] See Başar, Haurie and Ricci (1985), van der Ploeg and de Zeeuw (1990) and Rubio and Escriche (2001).

- the i-th player is endowed with the instantaneous payoff $\pi_i(\mathbf{x}(t), \mathbf{u}(t), t) \in C^2(X \times U \times [t_0, \infty))$, and is supposed to maximize the discounted objective functional:

$$J_i \equiv \int_{t_0}^{\infty} e^{-\rho_i(t-t_0)} \pi_i(\mathbf{x}(t), \mathbf{u}(t), t) dt \tag{2.1}$$

  subject to the kinematic equation:

$$\begin{cases} \dot{x}_s(t) = g_s(\mathbf{x}(t), \mathbf{u}(t), t) \\ x_s(t_0) = x_{s0} \end{cases} , \tag{2.2}$$

  where $g_s(\cdot) \in C^2(X \times U \times [t_0, \infty))$, $s = 1, \ldots, n$ and ρ_i is the constant force of interest for the i-th agent;[4]

- to satisfy the sufficiency condition for maximization, we assume that $\pi_i(\mathbf{x}(t), \mathbf{u}(t), t)$ is concave and $g_s(\mathbf{x}(t), \mathbf{u}(t), t)$ is quasi-concave in states and controls.

Definition 2.1. *A vector of strategies* $(u_1^*, \ldots, u_n^*) \in U$ *such that:*

$$J_i(u_1^*, \ldots, u_n^*) \geq J_i(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_n^*), \quad i = 1, \ldots, n \tag{2.3}$$

is:

1. *an open-loop Nash equilibrium if* u_i^* *depends on time* t *and on the given initial condition* $\mathbf{x}_0$ *(i.e. is an open-loop strategy for all* i*), and if* (2.3) *holds for all open-loop strategies* u_i*;*

2. *a feedback Nash equilibrium if* u_i^* *depends on* t, $\mathbf{x}$ *and if it is continuous in* t *and uniformly Lipschitz in* $\mathbf{x}$ *for each* t *(i.e. is a feedback strategy for all* i*), and if* (2.3) *holds for every possible initial condition* $(t_0, \mathbf{x}_0)$ *of* (2.2)*.*

Classically, open-loop equilibria are determined by applying Pontryagin's maximum principle, whereas optimal feedback trajectories request the solution of the Hamilton-Jacobi-Bellman equations, whenever that is possible.

[4]That is, we confine our attention to a setup with n players, n controls and n states, which is not as general as one could desire. In particular, it leaves out of the picture games treating common pool resource extraction, as well as environmental issues. The extension of the present setup to account for these aspects is left for future research.

2.1 Time consistency in simultaneous games

Definition 2.2 (Başar and Olsder, 1995, pp. 256-7; Dockner et al., 2000, p. 99 and p. 102). *An optimal control vector* $\mathbf{u}^*(\cdot) \in U$ *is called:*

1. *time consistent if its truncated part in the time interval* $[T, \infty)$*, where* $T > t_0$*, represents an equilibrium also for any subgame starting in* $t = T$*, given the vector of initial conditions* $\tilde{\mathbf{x}}(T)$*.*

2. *strongly time consistent if its truncated part in the time interval* $[T, \infty)$*, where* $T > t_0$*, represents an equilibrium also for any subgame starting in* $t = T$*, independently of the initial conditions* $\tilde{\mathbf{x}}(T)$*.*

By time consistency without further qualification we refer to what is often defined as weak time consistency, while strong time consistency corresponds to *subgame perfection*. If $\mathbf{u}^*(\cdot)$ is not subgame perfect, then it is a credible strategy vector under precommitment only, as shown by the classical following result:

Proposition 2.1. *Given an n-tuple of feedback strategies* $\widehat{\mathbf{u}}^*(\cdot) \in U$*, if the induced n-tuple of feedback strategies of every subgame starting in* T*, independently of the initial conditions* $\widetilde{x}(T)$*, can be played, then* $\widehat{\mathbf{u}}^*(\cdot)$ *is a strongly time consistent equilibrium.*

Proof: See Mehlmann (1988, pp. 65-67). ∎

State redundancy is easily detectable by applying Pontryagin's maximum principle: after solving the adjoint equations and substituting the costate variables thus found in the necessary conditions on controls, the resulting expression depends neither on the states nor on their initial values. The next proposition (e.g. Mehlmann and Willing, 1983) connects state-redundancy with time consistency:

Proposition 2.2. *If a differential game is state-redundant, then its open-loop Nash equilibrium is strongly time consistent.*

That is to say, if a game is state-redundant, its open-loop Nash equilibrium coincides with its feedback Nash equilibrium. State redundancy of open-loop Nash equilibria characterises trilinear games (Clemhout and Wan, 1974) and exponential games (Reinganum, 1982). A technique for identifying games whose open-loop equilibria are strongly time consistent is outlined by Fershtman (1987), who labels this class of games as *perfect games*. In general, any perfect game is either state-redundant or can be transformed into a state-redundant game (Mehlmann and Willing, 1983). For an exhaustive analysis of state redundancy, see Mehlmann (1988, ch. 4).

2.2 Time consistency in sequential games

Now consider the case of a two-player Stackelberg game, where $u_L(\cdot)$ and $u_F(\cdot)$ are respectively the control functions for the leader and for the follower, whose Hamiltonians are $H_L(u_L, u_F, \mathbf{x}, \lambda_L, t)$ and $H_F(u_F, u_L, \mathbf{x}, \lambda_F, t)$. The function

$$R_F(u_L, \mathbf{x}, \lambda_F, t) = \arg\max_{u_F} H_F(u_F, u_L, \mathbf{x}, \lambda_F, t)$$

where $R_F(u_L, \mathbf{x}, \lambda_F, t)$ denotes the follower's best response function to the leader's control path (see Dockner *et al.*, 2000, pp. 114-115).

Definition 2.3. *A pair of strategies* $(u_L^*, u_F^*) \in U$ *such that:*

$$J_L(u_L^*, u_F^*) \geq J_L(u_L, R_F(u_L, \mathbf{x}, \lambda_F, t)), \quad J_F(u_L^*, u_F^*) \geq J_F(u_L^*, u_F) \quad (2.4)$$

for all $u_L,\ u_F \in U$*, is:*

- *an open-loop Stackelberg equilibrium if* u_L^* *and* u_F^* *are open-loop strategies, and if* (2.4) *holds for all open-loop strategies* $u_L,\ u_F \in U$*;*
- *a feedback Stackelberg equilibrium if* u_L^* *and* u_F^* *are feedback strategies, and if* (2.4) *holds for every possible initial condition* $(t_0, \mathbf{x}_0)$ *of* (2.2)*.*

We know from Simaan and Cruz (1973a,b) that generally open-loop Stackelberg games yield time-inconsistent equilibria. In particular, if we call $\lambda_{Fs}^*(t)$ the follower's costate variables, we can give the following:

Definition 2.4. *If for every* $s = 1, \ldots, n$*,* $\lambda_{Fs}^*(t_0)$ *does not depend on the leader's control* $u_L(t)$*, then the initial value of the follower's costates is said to be uncontrollable by the leader. Otherwise, it is said to be controllable.*

Controllability leads to time inconsistency since the leader controls the follower's costates by manoeuvring $u_L(t)$ (cf. Xie, 1997; Dockner *et al.*, 2000). On the other hand, uncontrollability is a necessary condition to obtain strongly time consistent open-loop Stackelberg equilibria, but it is not sufficient, as we summarize in the following (which appears as Corollary 3 in Cellini, Lambertini and Leitmann, 2005, p. 188):

Proposition 2.3. *If a differential game is both uncontrollable by all of its players and state-redundant, then all of its open-loop Stackelberg equilibria are strongly time consistent.*

In the remainder, we intend to determine a suitable game structure whose properties are such that Stackelberg and Nash equilibria coincide under feedback information and therefore, provided Proposition 2.2 holds, the feedback Stackelberg equilibrium coincides with the open-loop Nash one. In so doing, we also extend the result obtained by Rubio (2006, Proposition 2.3, p. 210) for the two-player case and stationary solutions to a more general setup with n players partitioned in two groups, one of leaders and the other of followers and to the case of non-stationary solutions. Additionally, Rubio's interpretation of his own result is that "the feedback Stackelberg solution cannot generally be used to investigate leadership in the framework of a continuous-time differential game" (Rubio, 2006, p. 204). We take a different angle, so as to stress that indeed the property whereby feedback Stackelberg and open-loop Nash equilibria coincide, when it applies, allows us to solve strongly time-consistent hierarchical games using the simpler approach based on open-loop information and simultaneous play.

3 Additive separability and time consistency

Consider a game where a population of n players is divided into two groups, respectively formed by l and $n-l$ agents. Call:

- $\mathbf{u}_L = (u_{1L}, \ldots, u_{lL})$ the vector of controls of the first group of players (acting as leaders if the play is sequential);
- $\mathbf{u}_F = (u_{1F}, \ldots, u_{(n-l)F})$ the control variable vector of the second group of players (acting as followers if the play is sequential);
- $\mathbf{x}(t) = (\mathbf{x}_L, \mathbf{x}_F) = (x_{1L}(t), \ldots, x_{lL}(t), x_{1F}(t), \ldots, x_{(n-l)F}(t))$ the usual state variable vector,

all of them belonging to suitable open control and state sets. That is, we confine our attention to the case of n players, each being endowed with a single control, and n states. This game can be played either simultaneously or hierarchically; in the latter case, the first l agents form the leaders' group, whereas the remaining $n-l$ players represent the followers' group.

We assume that in both cases moves are simultaneous inside each of the two populations, i.e. within both groups a simultaneous competitive

Nash game is played. As usual, in the Stackelberg game case, the leaders incorporate the followers' first order conditions into their optimum problems before picking their own optimal controls.

Definition 3.1. *A differential game is additively separable if:*

1. *the i-th player is endowed with a payoff $\pi_i(\cdot)$ which is additively separable in states and controls, i.e.:*

$$\pi_i(\mathbf{x}(t), \mathbf{u}_L(t), \mathbf{u}_F(t), t) = \alpha_i(\mathbf{x}_L(t), \mathbf{u}_L(t), t) + \beta_i(\mathbf{x}_F(t), \mathbf{u}_F(t), t),$$

   for all $i = 1, \ldots, n$, where $\alpha_i(\cdot)$ and $\beta_i(\cdot)$ are concave and C^2 functions w.r.t. all of their respective arguments;

2. *the dynamic constraints are the kinematic equations:*

$$\begin{cases} \dot{x}_s(t) = L_s(\mathbf{x}_L(t), \mathbf{u}_L(t), t) + F_s(\mathbf{x}_F(t), \mathbf{u}_F(t), t) \\ x_s(t_0) = x_{s0} \end{cases}, \tag{3.1}$$

   $s = 1, \ldots, n$, where $L_s(\cdot)$ and $F_s(\cdot)$ are quasi-concave and C^2 functions w.r.t. all of their respective arguments.

This definition identifies those games where any multiplicative interplay among controls and states of players of either group is excluded, both in any player i's instantaneous payoff and in each state dynamics.[5] Nonetheless, within each group multiplicative effects among variables are admitted, provided the regularity requirements stated in the definition are satisfied. In such a game, the additive separability of dynamics and payoffs entails the following expression for the i-th player's present-value Hamiltonian:

$$H_i(\cdot) = e^{-\rho_i(t-t_0)} \Big[\alpha_i(\mathbf{x}_L, \mathbf{u}_L, t) + \beta_i(\mathbf{x}_F, \mathbf{u}_F, t) + \lambda_{ii}(t)(L_i(\mathbf{x}_L, \mathbf{u}_L, t) +$$

$$F_i(\mathbf{x}_F, \mathbf{u}_F, t)) + \sum_{s \neq i} \lambda_{is}(t)(L_s(\mathbf{x}_L, \mathbf{u}_L, t) + F_s(\mathbf{x}_F, \mathbf{u}_F, t))\Bigg],$$

[5]Our definition applies, e.g., to the sticky price oligopoly game with differentiated goods in Cellini and Lambertini (2007), with n players, each endowed with one control and one state. Other definitions of additive separability exist in the literature: for instance, Dockner *et al.* (1985, p. 188) define a class of differential games whose Hamiltonians are characterised by additive separability between the vector of states and the vector of controls, so that $\partial^2 H_i(\cdot)/\partial u_i \partial x_k = 0$ for all i, k. This clearly differs from our separability requirement.

where $\lambda_{is}(t)$ is the current value costate variable associated by player i with state variable x_s. On the basis of Definition 3.1, additive separability implies:

$$\frac{\partial^2 H_i(\cdot)}{\partial u_{jL}\partial u_{kF}} = 0, \; i = 1, \ldots, n, \quad j = 1, \ldots, l, \quad k = 1, \ldots, n-l.$$

I.e., the mixed second partial derivatives of all Hamiltonians vanish when they are calculated w.r.t. any two control variables belonging to different groups.

Now we proceed to compare both open-loop and feedback Nash as well as Stackelberg equilibria of such a game.

If the Hamiltonian functions are concave (and they are also C^2 by construction, in view of the above assumptions), the open-loop equilibria can be found by solving the necessary conditions (see Theorem 4.2 in Dockner *et al.*, 2000, pp. 93-94):

$$\frac{\partial H_j}{\partial u_{jL}} = 0 \iff \frac{\partial \alpha_j}{\partial u_{jL}} + \sum_{s=1}^{n} \lambda_{js} \frac{\partial L_j}{\partial u_{sL}} = 0, \;\; j = 1, \ldots, l. \tag{3.2}$$

$$\frac{\partial H_{k+l}}{\partial u_{kF}} = 0 \iff \frac{\partial \beta_{k+l}}{\partial u_{kF}} + \sum_{s=1}^{n} \lambda_{k+l,s} \frac{\partial F_{k+l}}{\partial u_{sF}} = 0, \;\; k = 1, \ldots, n-l. \tag{3.3}$$

In the next Proposition we shall call $V^i(\mathbf{x}, t)$ the i-th optimal value function and

$$Z_i(\cdot) = \alpha_i(\cdot) + \beta_i(\cdot) + \sum_{s=1}^{n} \frac{\partial V^i(\cdot)}{\partial x_s}[L_s(\cdot) + F_s(\cdot)], \quad i = 1, \ldots, n. \tag{3.4}$$

Proposition 3.1. *Suppose that in an additively separable game respecting Definition 3.1, the functions Z_i is C^2 and concave w.r.t. u_i for all $i = 1, \ldots, n$. Then the feedback Nash equilibrium and the feedback Stackelberg equilibrium are identical.*

Proof: First, consider the game played simultaneously among all n players, without any division into groups. There exists a feedback Nash equilibrium $\mathbf{u}^*$ whose i-th component maximizes the right hand side of the Hamilton-Jacobi-Bellman equation:

$$-\frac{\partial V^i}{\partial t} + \rho_i V^i = \max_{u_i} Z_i, \tag{3.5}$$

for all $i = 1, \ldots, n$, and such that

$$\frac{\partial Z_{i \;|\; u_i = u_i^*}}{\partial u_i} = 0, \qquad Hess(Z_{i \;|\; u_i = u_i^*}) \text{ negative definite,} \tag{3.6}$$

where $Hess(\cdot)$ denotes the Hessian matrix. Note that by the additively separable structure of the game at hand, the first order conditions on Z_j, $j = 1, \ldots, l$, involve only the control variables $u_1, \ldots, u_l$, i.e.:

$$\frac{\partial Z_j}{\partial u_j} = \frac{\partial \alpha_j}{\partial u_j} + \sum_{s=1}^{n} \frac{\partial V^j(\cdot)}{\partial x_s} \frac{\partial L_s}{\partial u_j} = 0 \tag{3.7}$$

so that

$$\frac{\partial u_j^*}{\partial u_k} = 0, \qquad \text{if } \ j \in \{1, \ldots, l\} \quad \text{and} \quad k \in \{l+1, \ldots, n\}. \tag{3.8}$$

Likewise, the $n-l$ remaining first order conditions contain only $u_{l+1}, \ldots, u_n$, i.e.:

$$\frac{\partial Z_k}{\partial u_k} = \frac{\partial \beta_k}{\partial u_k} + \sum_{s=1}^{n} \frac{\partial V^k(\cdot)}{\partial x_s} \frac{\partial F_s}{\partial u_k} = 0 \tag{3.9}$$

so that

$$\frac{\partial u_k^*}{\partial u_j} = 0, \qquad \text{if } \ j \in \{1, \ldots, l\} \quad \text{and} \quad k \in \{l+1, \ldots, n\}. \tag{3.10}$$

Alternatively, the same game is played sequentially among the group of the first l players, acting as leaders, and the group of the remaining $n-l$ players, acting as followers. Within each of the two sets of players the game is simultaneous. As usual, we proceed by backward induction considering first the maximum problem of a generic follower.

By the concavity hypothesis on Z_k, there exists a vector of strategies $\mathbf{u}_F^* = (u_{l+1,F}^*, \ldots, u_{n,F}^*)$ whose j-th component maximizes the right hand side of (3.5) for $k = l+1, \ldots, n$ and such that

$$\frac{\partial Z_{k \ | \ u_k = u_{k,F}^*}}{\partial u_k} = 0, \qquad\qquad Hess(Z_{k \ | \ u_k = u_{k,F}^*}) \text{ negative definite.} \tag{3.11}$$

This exactly replicates the conditions in (3.6), and in particular the set of first order conditions is the same as (3.9). Therefore, $u_{k,F}^* = u_k^*$ for all $k = l+1, \ldots, n$.

By (3.10), the followers' optimal strategies $\mathbf{u}_F^*$ do not depend on any leaders' controls. Evaluating the leaders' functions Z_j in $\mathbf{u}_F^*$, we notice that only the functions $\beta_j(\cdot)$ and $F_j(\cdot)$ are affected by that substitution.

Then, plugging $\mathbf{u}_F^*$ into (3.5), and solving for the leaders' group, the derivatives of all $\beta_j(\cdot)$ and $F_j(\cdot)$ w.r.t. the leaders' controls vanish, so the conditions for maximization are the same as in the previous game. As in the first case, there exists a vector of strategies $\mathbf{u}_L^* = (u_{1,L}^*, \ldots, u_{l,L}^*)$ such that Z_j is maximized, and

$$\frac{\partial Z_{j \ | \ u_j = u_{j,L}^*}}{\partial u_j} = 0, \qquad\qquad Hess(Z_{j \ | \ u_j = u_{j,L}^*}) \text{ negative definite,} \tag{3.12}$$

for all $j \in \{1, \ldots, l\}$. Since such relations exactly coincide with (3.6) in the totally simultaneous case, then the set of first order conditions for the leaders coincide with (3.7), yielding $u_{j,L}^* = u_j^*$ for all $j = 1, \ldots, l$. Then we can conclude that the feedback Nash equilibrium and the feedback Stackelberg equilibrium coincide in this class of games. ■

Note that the previous result entails that Nash and Stackelberg feedback solutions are observationally equivalent and therefore, *ex post*, one could not tell whether the vector of optimal controls is the outcome of sequential rather than simultaneous play. Moreover, when such a game is a Stackelberg one, it turns out to be uncontrollable by all players. Proposition 2.2, ensuring strong time consistency for open-loop Nash equilibria of a state-redundant game, and Proposition 3.1, implying that feedback Nash and feedback Stackelberg equilibria coincide, jointly entail the following:

Corollary 3.1. *If an additively separable game is state-redundant and uncontrollable, then its open-loop Nash and feedback Stackelberg equilibria coincide and they are all strongly time consistent.*

It is worth mentioning briefly that an additively separable game which is state-redundant but controllable, produces feedback Stackelberg and Nash equilibria that coincide with the open-loop Nash equilibrium, while clearly the open-loop Stackelberg equilibrium is time inconsistent.

4 An application to an advertising model

We examine a modified version of the game by Leitmann and Schmitendorf (1978) and Feichtinger (1983), built up in such a way to meet Definition 3.1. Consider an oligopoly where the individual market share changes over time as a function of firms' advertising investments and decreases over time at a fixed rate $\delta \geq 0$:[6]

$$\dot{x}_i(t) = u_i(t) + s \sum_{j \neq i} u_j(t) - \delta x_i(t). \tag{4.1}$$

[6] As anywhere else in the paper, also in this example we assume that states and controls belong to open sets. However, this example yields a single inner solution.

Note that (4.1) features an externality from the control of each j onto the state of i but there is no direct interplay between any u_j and x_i; parameter $s \in [-1, 1]$ measures the spillover effect, as in the static version. The instantaneous profit of firm i is $\pi_i = px_i - bu_i^2$, where p and b are positive constants.[7] Denoting by $\rho \geq 0$ the intertemporal discount rate common to all firms, each firm has to determine the amount of advertising that solves the following problem

$$\max_{u_i} J_i^{LS} \equiv \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \pi_i dt,$$

subject to (4.1). If we consider two groups of agents (l leaders and $n - l$ followers), this game is additively separable in the sense of Definition 3.1. In fact, by sticking to the notation of Section 3:

- If the i-th player is a leader, i.e. $1 \leq i \leq l$,

$$\alpha_i(\cdot) = px_i - bu_i^2, \qquad \beta_i(\cdot) = 0,$$

$$L_i(\cdot) = s\sum_{j=1}^{l} u_j + (1-s)u_i - \delta x_i, \qquad F_i(\cdot) = s\sum_{j=l+1}^{n} u_j.$$

- If the i-th player is a follower, i.e. $l + 1 \leq i \leq n$,

$$\alpha_i(\cdot) = 0, \qquad \beta_i(\cdot) = px_i - bu_i^2,$$

$$L_i(\cdot) = s\sum_{j=1}^{l} u_j, \qquad F_i(\cdot) = s\sum_{j=l+1}^{n} u_j + (1-s)u_i - \delta x_i.$$

The i-th current value Hamiltonian is:

$$H_i(\cdot) = px_i - bu_i^2 + \lambda_{ii}\left(u_i + s\sum_{j\neq i} u_j - \delta x_i\right) + \sum_{j\neq i} \lambda_{ij}\left(u_j(t) + s\sum_{k\neq j} u_k - \delta x_j\right).$$

First, we search for the open-loop solution. The necessary conditions are the following:

$$\frac{\partial H_i}{\partial u_i} = -2bu_i + \lambda_{ii} + s\sum_{j\neq i} \lambda_{ij} = 0 \tag{4.2}$$

[7] The main difference between between the present model and Leitmann and Schmitendorf (1978) is that here the state kinematics is additively separable in states and controls.

$$\dot{\lambda}_{ii} = \rho\lambda_{ii} - \frac{\partial H_i}{\partial x_i} = (\rho+\delta)\lambda_{ii} - p \tag{4.3}$$

$$\dot{\lambda}_{ij} = \rho\lambda_{ij} - \frac{\partial H_i}{\partial x_j} = (\rho+\delta)\lambda_{ij} \tag{4.4}$$

for all $i \neq j$, $i, j = 1, \ldots, n$.

Equations (4.4) can be integrated by separation of variables and admit the solutions $\lambda_{ij} \equiv 0$ when $\lambda_{ij}(t_0) = 0$.

Through standard calculations with initial conditions $u_i(t_0) = u_{i0}$ we obtain the expression for the open-loop equilibrium strategy for each player:

$$u_i^*(t) = \left[u_{i0} - \frac{p}{2(\rho+\delta)b}\right] e^{(\rho+\delta)t} + \frac{p}{2(\rho+\delta)b} \tag{4.5}$$

Since all equations (4.3) are decoupled, the reaction functions are all orthogonal. Thus $(u_1^*, \ldots, u_n^*)$ is both an open-loop Nash and Stackelberg equilibrium.

Note that, by substitution, we can also compute the related optimal states by solving the linear Cauchy problem:

$$\begin{cases} \dot{x}_i(t) = -\delta x_i(t) + \left[u_{i0} + s\sum_{j\neq i} u_{j0} - \frac{p}{2(\rho+\delta)b}(1+s(n-1))\right] e^{(\rho+\delta)t} \\ x_i(t_0) = x_{i0} > 0 \end{cases},$$

whose solutions are:

$$x_i^*(t) = x_{i0}e^{-\delta t} + \left[u_{i0} + s\sum_{j\neq i} u_{j0} - \frac{p}{2(\rho+\delta)b}(1+s(n-1))\right] \frac{e^{(\rho+\delta)t}}{\rho+2\delta}. \tag{4.6}$$

A sufficient condition for the positivity of the state at all times is:

$$p \in \left(0, \frac{2b(\rho+\delta)\left(u_{i0} + s\sum_{j\neq i} u_{j0}\right)}{1+s(n-1)}\right). \tag{4.7}$$

As far as feedback strategies are concerned, consider the i-th optimal value function $V^i(x,t) = A(t) + B(t)x_i$. This is linear in the i-th state variable and with two time-dependent coefficients to be determined (for a typical example of this technique, see Dockner *et al.*, 2000, pp. 53-58). The sufficient conditions for maximization imply:

$$u_i = \frac{1}{2b}\frac{\partial V^i}{\partial x_i} = \frac{B(t)}{2b}. \tag{4.8}$$

Plugging (4.8) into the i-th Hamilton-Jacobi-Bellman equation we obtain:

$$-\dot{A}(t)-\dot{B}(t)x_i+\rho A(t)+\rho B(t)x_i=px_i+\left(\frac{1}{2}+s(n-1)\right)\frac{B^2(t)}{2b}-\delta B(t)x_i,$$

so that $A(t)$ and $B(t)$ can be determined by solving the following:

$$\begin{cases}\dot{A}(t)=\rho A(t)-\left(\frac{1}{2}+s(n-1)\right)\frac{B^2(t)}{2b} \\ \dot{B}(t)=(\rho+\delta)B(t)-p\end{cases},$$

i.e.

$$A(t)=A(t_0)e^{\rho t}-\frac{1}{2b}\left(\frac{1}{2}+s(n-1)\right)\left[\left(B(t_0)-\frac{p}{\rho+\delta}\right)^2\frac{e^{2(\rho+\delta)t}}{\rho+2\delta}+\right.$$

$$\left.+\frac{2p}{(\rho+\delta)\delta}\left(B(t_0)-\frac{p}{\rho+\delta}\right)e^{(\delta+\rho)t}-\frac{p^2}{\rho(\delta+\rho)^2}\right],$$

$$B(t)=\left(B(t_0)-\frac{p}{\rho+\delta}\right)e^{(\rho+\delta)t}+\frac{p}{\rho+\delta}.$$

It can be easily verified that this game is state-redundant and furthermore no player has control over other players' strategies. Consequently Proposition 3.3 can be applied to this game, whose open-loop Nash and feedback Stackelberg equilibria coincide. The value added of this kind of structure is that one can grasp the essence of the model by solving the easiest of all possible cases, which is the open-loop Nash equilibrium.

5 Concluding Remarks

We have revisited the issue of time (in)consistency of hierarchical differential games, introducing a specific definition of additive separability of payoff functions and state dynamics w.r.t. states and controls. Our notion of additive separability, in combination with state-redundancy, implies that the feedback Stackelberg equilibrium and the open-loop Nash equilibrium coincide. This is due to the fact that additive separability entails the coincidence between feedback Stackelberg and Nash equilibria, the latter collapsing onto the open-loop Nash solution due to state redundancy. This offers, at least in the class of games we have identified, a way out of two well known problems: (i) the time inconsistency issue that usually obtains in open-loop Stackelberg games; and (ii) the lack of mathematical tools for the analytical solution of feedback Stackelberg games.

References

[1] T. Başar and G.J. Olsder (1995^2), *Dynamic Noncooperative Game Theory*. Academic Press, San Diego.

[2] T. Başar, A. Haurie and G. Ricci (1985), "On the dominance of capitalism leadership in a feedback-Stackelberg solution of a differential game model of capitalism", *Journal of Economic Dynamics and Control*, **9**. pp. 101-125.

[3] R. Cellini and L. Lambertini (2007), "A Differential Oligopoly Game with Differentiated Goods and Sticky Prices", *European Journal of Operational Research*, **176**, 1131-1144.

[4] R. Cellini, L. Lambertini and G. Leitmann (2005), "Degenerate Feedback and Time Consistency in Differential Games", in E.P. Hofer and E. Reithmeier (eds), *Modeling and Control of Autonomous Decision Support Based Systems. Proceedings of the 13th International Workshop on Dynamics and Control*, Aachen, Shaker Verlag, 185-192.

[5] S. Clemhout and H.Y. Wan, Jr. (1974), "A class of trilinear differential games", *Journal of Optimization Theory and Applications*, **14**, pp. 419-424.

[6] E.J. Dockner, G. Feichtinger, S. Jørgensen (1985), "Tractable classes of nonzero-sum open-loop Nash differential games: theory and examples", *Journal of Optimization Theory and Applications*, **45**, pp. 179-197.

[7] E.J. Dockner, S. Jørgensen, N.V. Long and G. Sorger (2000), *Differential Games in Economics and Management Science*. Cambridge University Press, Cambridge.

[8] G. Feichtinger (1983), "The Nash solution of an advertising differential game: generalization of a model by Leitmann and Schmitendorf", *IEEE Transactions on Automatic Control*, **28**, pp. 1044-1048.

[9] C. Fershtman (1987), "Identification of classes of differential games for which the open loop is a degenerate feedback Nash equilibrium ", *Journal of Optimization Theory and Applications*, **55**, pp. 217-231.

[10] G. Leitmann and W.E. Schmitendorf (1978), "Profit maximization through advertising: a nonzero sum differential game approach", *IEEE Transactions on Automatic Control*, **23**, pp. 646-650.

[11] A. Mehlmann (1988), *Applied Differential Games*, New York, Plenum Press.

[12] A. Mehlmann and R. Willing (1983), "On nonunique closed-loop Nash equilibria for a class of differential games with a unique and degenrate feedback solution ", *Journal of Optimization Theory and Applications*, **41**, pp. 463-472.

[13] J.F. Reinganum (1982), "A class of differential games for which the Closed-Loop and Open-Loop Nash equilibria coincide ", *Journal of Optimization Theory and Applications*, **36**, pp. 253-262.

[14] S.J. Rubio (2006), "On coincidence of feedback Nash equilibria and Stackelberg equilibria in economic applications of differential games", *Journal of Optimization Theory and Applications*, **128**, pp. 203-221.

[15] S.J. Rubio and L. Escriche (2001), "Strategic Pigouvian taxation, stock externalities and polluting non-renewable resources ", *Journal of Public Economics*, **79**, 2, pp. 297-313.

[16] M. Simaan and J. B. Cruz Jr. (1973a), "On the Stackelberg strategy in non-zero sum games", *Journal of Optimization Theory and Applications*, **11**, pp. 533-555.

[17] M. Simaan and J. B. Cruz Jr. (1973b),"Additional aspects of the Stackelberg strategy in non-zero sum games", *Journal of Optimization Theory and Applications*, **11**, pp. 613-626.

[18] F. van der Ploeg and A.J. de Zeeuw (1990), "Perfect equilibrium in a model of competitive arms accumulation", *International Economic Review*, **31**, pp. 131-146.

[19] D. Xie (1997), "On time inconsistency: a technical issue in Stackelberg differential games", *Journal of Economic Theory*, **76**, 2, pp. 412-430.