The Leitmann-Schmitendorf advertising game with $n$ players and time discounting

Davide Dragone Luca Lambertini and Arsen Palestini

*Department of Economics, University of Bologna
Strada Maggiore 45, 40125 Bologna, Italy
davide.dragone@unibo.it; luca.lambertini@unibo.it; palestini@math.unifi.it

Abstract

The extension of the Leitmann-Schmitendorf advertising game to $n$ players and positive time discounting is investigated. We show that the strong time consistency of the open-loop Nash equilibrium is preserved. As to optimal controls, while the boundary solution is unaffected by the number of firms as well as discounting, the inner solution depends on industry structure. The fully symmetric version of the game allows us to identify the parameter regions wherein both solutions are sustainable.

Keywords: differential games, advertising, open-loop control

*We thank two anonymous referees and Silvano Baggio for useful comments and suggestions. The usual disclaimer applies.
1 Introduction

The analysis of dynamic games in marketing is a long-standing and lively stream of research.\(^1\) One aspect that has received a large amount of attention is the incentive for firms to invest in advertising campaigns in order to increase their respective market shares. Here, we revisit the advertising game by Leitmann and Schmitendorf (1978),\(^2\) which originally only involved 2 players, to extend it to the case of \( n \) firms and discounting.

We characterise the inner and boundary equilibria generated by the game, and assess their respective stability properties. Then, we confine our attention to the fully symmetric setup and evaluate the sustainability of the two steady state equilibria by imposing non-negativity constraints on optimal controls and states, as well as profits. By doing so, we establish that (i) the sustainability of the inner solution depends on market affluency but not on the number of firms, while (ii) the admissibility of the boundary solution depends on a condition involving both market affluency and industry structure. Finally, in the region where both solutions are sustainable, we find that their relative profitability is determined by the relative size of discounting and the decay rate of firms’ market shares.

The remainder of the paper is structured as follows. The setup is laid out in section 2. Equilibria are characterised in section 3. Section 4 contains the analysis of the fully symmetric game. Concluding remarks are in section 5.

2 The setup

Our setup closely reflects Leitmann and Schmitendorf (1978) and Feichtinger (1983), except for the number of players and discounting. Consider \( n \geq 2 \) firms competing in an advertising differential game over the finite horizon \([0, T]\). Each firm sells a substitutable commodity and aims to maximize her own profit. Call \( p_i > 0 \) the revenue rate of the \( i \)-th agent when the related product conquers the whole market. The state variable \( x_i(t) \) denotes the fraction of the total market buying from firm \( i \) at time instant \( t \in [0, T] \).

\(^{1}\)For exhaustive surveys, see Feichtinger et al. (1994), Dockner et al. (2000) and Jørgensen and Zaccour (2004).

\(^{2}\)See also Feichtinger (1983).
The state space $X$ is defined as follows:

$$X = \left\{ (x_1(t), \ldots, x_n(t)) \in \mathbb{R}_+^n : \sum_{i=1}^n x_i(t) \leq 1 \right\}.$$

The strategic variables of the players are $u_i(t)$, representing the $i$-th advertising effort rate at time $t$. Given $n$ constants $c_1, \ldots, c_n \in \mathbb{R}_+$, $u_i(t) \in U_i := [0, \frac{1}{c_i}]$, i.e., the strategy space is a compact set for every $i = 1, \ldots, n$.

Within this framework, the $i$-th firm faces the problem of maximizing the objective functional

$$J_i = \int_0^T e^{-\rho_i t}[p_i x_i(t) - u_i(t)]dt$$

subject to the kinematic equations:

$$\dot{x}_i(t) = u_i(t) - \frac{c_i u_i^2}{2} - k_i x_i(t) \left( \sum_{j \neq i} u_j(t) \right) - \beta_i x_i(t),$$

for all $i = 1, \ldots, n$.

The initial conditions of (2.2) are given: $x_i(0) = x_{i0} \geq 0$ for all $i$. In (2.2) $\beta_1, \ldots, \beta_n$ are appropriate positive constants indicating the decay rates of the market shares of the firms, whereas $k_1, \ldots, k_n$ are positive parameters quantifying the absorptive capacity in the interaction term.

The most important novelty of this formulation with respect to the original 2-firm model is the absorptive term in (2.2): the dynamic expansion of the $i$-th portion of the market is negatively influenced by all the remaining players’ strategic actions in the same way.

In addition, it is worth mentioning that in the original version no intertemporal discount rate appears, as it is common not to consider any discounting of future payoffs in finite time. Conversely, Cellini et al. (2005) investigate the same game endowed with a discounted functional. In the remainder, we let $\rho_1 = \ldots = \rho_n := \rho$ constant for all players.

In the following, the time argument will be dropped for brevity. The $i$-th firm’s current-value Hamiltonian writes as follows:

$$H_i(\cdot) = p_i x_i - u_i + \lambda_{ii} \left( u_i - \frac{c_i u_i^2}{2} - k_i x_i \sum_{j \neq i} u_j - \beta_i x_i \right) +$$

(2.3)
where $\lambda_{ij}(\cdot)$ is the costate variable associated by the $i$-th player to the $j$-th transition function.

Pontryagin’s Maximum Principle yields the following $n$ first order conditions:

$$\frac{\partial H_i}{\partial u_i} = -1 + \lambda_{ii} - c_i u_i \lambda_{ii} - k_j x_j \lambda_{ij} = 0.$$  

(2.5)

The adjoint equations and the related transversality conditions, respectively, read as:

$$\dot{\lambda}_{ii} = \rho \lambda_{ii} - \frac{\partial H_i}{\partial x_i} = \rho \lambda_{ii} - p_i + \beta_i \lambda_{ii} + \lambda_{ii} \sum_{m \neq i} u_m \quad \forall i,$$

(2.6)

$$\dot{\lambda}_{ij} = \rho \lambda_{ij} - \frac{\partial H_i}{\partial x_j} = \rho \lambda_{ij} + \beta_j \lambda_{ij} + \lambda_{ij} k_j \sum_{l \neq j} u_l \quad \forall j \neq i,$$

(2.7)

and

$$\lambda_{ij}(T)x_i(T) = 0,$$

(2.8)

for all $i, j = 1, \ldots, n$.

Equations (2.7) together with (2.8) can be trivially integrated by separation of variables and admit the solutions $\lambda_{ij} \equiv 0$. Using this result to simplify (2.5) suffices to prove that the open-loop Nash equilibrium is a degenerate feedback one, as in the original two-player version.

The first order conditions (2.5) imply the following nonlinear relations between costates and controls and their respective first order derivatives w.r.t. time:

$$\lambda_{ii} = \frac{1}{1 - c_i u_i} \iff \dot{\lambda}_{ii} = \frac{c_i \dot{u}_i}{(1 - c_i u_i)^2}.$$  

(2.9)

and by substitution in (2.6) we obtain the following state-control system:

$$\begin{cases}
    \dot{x}_i = u_i - \frac{c_i u_i^2}{2} - k_i x_i \left( \sum_{j \neq i} u_j \right) - \beta_i x_i \\
    \dot{u}_i = \frac{1}{c_i} \left[ (1 - c_i u_i) \left( \rho + \beta_i + k_i \sum_{j \neq i} u_j \right) - p_i (1 - c_i u_i)^2 \right]
\end{cases}.$$  

(2.10)

Before proceeding with the equilibrium analysis, we can immediately note that the control equations in (2.10) do not involve any state variable, hence they provide open-loop optimal Nash equilibria of the game.
3 Equilibrium points of the system

We first consider a non-symmetric case. The steady states of the model can be separately investigated as follows:

**Proposition 3.1.** The dynamic system (2.10) admits the boundary equilibrium point $P_1 = (x_1^*, \ldots, x_n^*, u_1^*, \ldots, u_n^*)$, where:

$$x_i^* = \frac{1}{2c_i \left( \beta_i + k_i \sum_{j \neq i} \frac{1}{c_j} \right)}, \quad u_i^* = \frac{1}{c_i},$$

for all $i = 1, \ldots, n$, which is feasible for our problem if:

$$\sum_{l=1}^{n} \frac{1}{c_l \left( \beta_l + k_l \sum_{j \neq l} \frac{1}{c_j} \right)} \leq 2. \tag{3.1}$$

**Proof.** The steady states of the dynamic system are the solutions of the following nonlinear system:

$$\begin{cases} u_i - \frac{c_i u_i^2}{2} - k_i x_i \left( \sum_{j \neq i} u_j \right) - \beta_i x_i = 0 \\ (1 - c_i u_i) \left( \rho + \beta_i + k_i \sum_{j \neq i} u_j \right) - p_i (1 - c_i u_i)^2 = 0 \end{cases}. \tag{3.2}$$

The first one immediately calculable is the boundary solution given by:

$$x_i^* = \frac{1}{2c_i \left( \beta_i + k_i \sum_{j \neq i} \frac{1}{c_j} \right)}, \quad u_i^* = \frac{1}{c_i},$$

which is feasible when the sum of all $x_i^*$ does not exceed one, i.e. when (3.1) holds. \qed

Note that the state and control at the boundary equilibrium are indeed independent on the discount rate $\rho$. In order to characterise the inner solution, now we consider $0 \leq u_i < \frac{1}{c_i}$ for all $i$. 


Proposition 3.2. If $k_i > p_i c_i$ for all $i = 1, \ldots, n$ then (2.10) admits an additional inner equilibrium point $P_2 = (\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{u}_1, \ldots, \tilde{u}_n)$, where:

$$
\tilde{x}_i = \frac{2\tilde{u}_i - c_i \tilde{u}_i^2}{2(\beta_i + k_i \sum_{j \neq i} \tilde{u}_j)}, \quad (3.3)
$$

$\forall i = 1, \ldots, n$;

$$
\tilde{u}_i = \frac{k_i}{k_i - p_i c_i} \left[ \frac{p_n - \beta_n - \rho}{k_n} - \frac{p_i - \beta_i - \rho}{k_i} + \left( 1 - \frac{p_n c_n}{k_n} \right) \tilde{u}_n \right], \quad (3.4)
$$

$\forall i = 1, \ldots, n$;

$$
\tilde{u}_n = \frac{p_n - \beta_n - \rho}{k_n} - \sum_{j=1}^{n-1} \left[ \frac{k_j}{k_j - p_j c_j} \left( \frac{p_n - \beta_n - \rho}{k_n} - \frac{p_j - \beta_j - \rho}{k_j} \right) \right] k_n. \quad (3.5)
$$

$P_2$ is feasible if and only if $0 \leq c_i \tilde{u}_i < 1$ for all $i = 1, \ldots, n$ and $(\tilde{x}_1, \ldots, \tilde{x}_n) \in X$.

Proof. The $n$ linear control equations

$$
\rho + \beta_i + k_i \sum_{j \neq i} u_j - p_i (1 - c_i u_i) = 0
$$

can also be arranged in the following matrix form:

$$
\begin{pmatrix}
\frac{p_1 c_1}{k_1} & 1 & 1 & \ldots & 1 \\
1 & \frac{p_2 c_2}{k_2} & 1 & \ldots & 1 \\
1 & 1 & \frac{p_3 c_3}{k_3} & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1 & \frac{p_n c_n}{k_n}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
\vdots \\
u_n
\end{pmatrix}
=
\begin{pmatrix}
p_1 - \beta_1 - \rho \\
p_2 - \beta_2 - \rho \\
p_3 - \beta_3 - \rho \\
\vdots \\
p_n - \beta_n - \rho
\end{pmatrix}. \quad (3.6)
$$

Any linear system

$$
\begin{pmatrix}
a_1 & 1 & 1 & \ldots & 1 \\
1 & a_2 & 1 & \ldots & 1 \\
1 & 1 & a_3 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1 & a_n
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
\vdots \\
u_n
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
\vdots \\
b_n
\end{pmatrix} \quad (3.7)
$$
admits the unique solution:

\[
\tilde{u}_l = \frac{b_n - b_l - (a_n - 1)\tilde{u}_n}{1 - a_l}, \quad 1 \leq l < n, \tag{3.8}
\]

\[
\tilde{u}_n = \frac{b_n - \sum_{j=1}^{n-1} \frac{b_n - b_j}{1 - a_j}}{a_n - (a_n - 1) \sum_{j=1}^{n-1} \frac{1}{1 - a_j}}, \tag{3.9}
\]

if \(a_i < 1\) for all \(i\).

In our case, by taking \(a_i = \frac{p_i c_i}{k_i}\) and \(b_i = \frac{p_1 - \beta_i - \rho}{k_i}\) we easily obtain the solution given by (3.4) - (3.5). Then, by plugging them into the first equation of (3.2) we determine (3.3).

It is worth stressing that the inner solution \(P_2\) does depend on both industry structure (the number of firms) and the discount rate.

As usual, the stability analysis involves an investigation on the eigenvalues of the \(2n \times 2n\) Jacobian matrix of (2.10):

\[
J = \begin{pmatrix}
\frac{\partial \dot{x}_1}{\partial x_1} & \ldots & \frac{\partial \dot{x}_1}{\partial x_n} & \frac{\partial \dot{x}_1}{\partial u_1} & \ldots & \frac{\partial \dot{x}_1}{\partial u_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \dot{x}_n}{\partial x_1} & \ldots & \frac{\partial \dot{x}_n}{\partial x_n} & \frac{\partial \dot{x}_n}{\partial u_1} & \ldots & \frac{\partial \dot{x}_n}{\partial u_n} \\
\frac{\partial \dot{u}_1}{\partial x_1} & \ldots & \frac{\partial \dot{u}_1}{\partial x_n} & \frac{\partial \dot{u}_1}{\partial u_1} & \ldots & \frac{\partial \dot{u}_1}{\partial u_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \dot{u}_n}{\partial x_1} & \ldots & \frac{\partial \dot{u}_n}{\partial x_n} & \frac{\partial \dot{u}_n}{\partial u_1} & \ldots & \frac{\partial \dot{u}_n}{\partial u_n}
\end{pmatrix}, \tag{3.10}
\]

in which the only first order derivatives which are not identically zero are:

\[
\frac{\partial \dot{x}_i}{\partial x_i} = -k_i \sum_{j \neq i} u_j - \beta_i, \quad \frac{\partial \dot{x}_i}{\partial u_i} = 1 - c_i u_i,
\]

\[
\frac{\partial \dot{u}_i}{\partial u_i} = 2p_i(1 - c_i u_i) - \rho - k_i \sum_{j \neq i} u_j - \beta_i,
\]

for all \(i = 1, \ldots, n\), and

\[
\frac{\partial \dot{x}_i}{\partial u_j} = -k_i x_i, \quad \frac{\partial \dot{u}_i}{\partial u_j} = \frac{1}{c_i}(1 - c_i u_i)k_i
\]
for all $i \neq j$.

Such peculiar characteristics make it possible for us to confine our analysis to the north-west and to the south-east $n \times n$ submatrices of $J$. The first $n$ diagonal entries are eigenvalues of $J$ for any possible equilibrium point, and their negativity ensures that no unstable node can occur.

**Proposition 3.3.** The steady state $P_1$ is a globally asymptotically stable equilibrium point for (2.10).

*Proof.* (3.10) evaluated at $P_1$ is an upper triangular matrix whose principal diagonal entries are:

$$-k_1 \sum_{j \neq 1} \frac{1}{c_j} - \beta_1, \ldots, -k_n \sum_{j \neq n} \frac{1}{c_j} - \beta_n, -\rho - k_1 \sum_{j \neq 1} \frac{1}{c_j} - \beta_1, \ldots, -\rho - k_n \sum_{j \neq n} \frac{1}{c_j} - \beta_n,$$

which are also the eigenvalues of $J(P_1)$. Since they are all strictly negative, $P_1$ is a global attractor for the system (2.10). \hfill \Box

**Proposition 3.4.** When it exists, the steady state $P_2$ is a saddle point equilibrium for (2.10) if

$$\frac{\partial \tilde{u}_i}{\partial u_i}(\tilde{u}_1, \ldots, \tilde{u}_n) = 2p_i (1 - c_i \tilde{u}_i) - \rho - k_i \sum_{j \neq i} \tilde{u}_j - \beta_i > 0 \quad \forall \ i = 1, \ldots, n.$$

*Proof.* The Jacobian matrix evaluated at $P_2$ has the following form:

$$J(P_2) = \begin{pmatrix} A & B \\ - & - & - & - & - & - & - & - \\ 0 & C \end{pmatrix}, \quad (3.11)$$

where $A$ is diagonal with all negative entries $-k_i \sum_{j \neq i} \tilde{u}_j - \beta_i, i = 1, \ldots, n$, $0$ is the null $n \times n$ matrix, hence the qualitative nature of $P_2$ can be determined by restricting our analysis to the matrix $C = (c_{ij})_{i,j=1,\ldots,n}$, where

$$c_{ii} = 2p_i (1 - c_i \tilde{u}_i) - \rho - k_i \sum_{j \neq i} \tilde{u}_j - \beta_i, \quad \forall \ i = 1, \ldots, n,$$
\[ c_{ij} = \frac{(1 - c_i \tilde{u}_i) k_i}{c_i}, \quad \forall i \neq j. \]

The constraint on the control variables entails that all \( c_{ij} \) are positive; therefore if all the remaining entries of \( C \) are positive as well, by the Perron-Frobenius theorem (see, for instance, MacCluer, 2000) \( C \) admits a positive real eigenvalue, ensuring that \( P_2 \) is a saddle point. \( \Box \)

## 4 The fully symmetric game

In this section, we will exploit a symmetry hypothesis on the system (2.10) in order to exactly determine the open-loop (and closed-loop) Nash trajectories. Recall that since the game is a linear state one, such hypothesis could also be removed for this calculation, which would indeed become tedious. Set \( p_i = p_j := p, c_i = c_j := c, k_i = k_j := k, \beta_i = \beta_j := \beta, \) and \( x_i = x_j := x, u_i = u_j := u. \) The resulting symmetrized system is:

\[
\begin{aligned}
\dot{x} &= u - \frac{cu^2}{2} - (n - 1)ku - \beta x, \\
\dot{u} &= \frac{1}{c}[(1 - cu) (\rho + \beta + (n - 1)ku) - p(1 - cu)^2]
\end{aligned}
\]  

(4.1)

with the associated initial conditions \( x(0) = x_0, u(0) = u_0. \) The optimal control \( \pi(t) \) can be found by integrating the second equation of (4.1) by separation of variables:

\[
\pi(t) = \frac{\rho + \beta - p + [pc + (n - 1)k]u_0 e^{(\rho + \beta + \frac{(n - 1)k}{c})t} - \rho - \beta + p}{\frac{1}{1-cu_0} e^{(\rho + \beta + \frac{(n - 1)k}{c})t} + pc + (n - 1)k}
\]  

(4.2)

and consequently, by substitution, the related optimal state trajectory \( \pi(t) \) turns out to be:

\[
\pi(t) = \left( x_0 + \int_0^t \left( \pi(s) - \frac{\pi^2(s)}{2} \right) e^{\beta s + k(n + 1) \int_0^s \pi(r) dr} ds \right) e^{\beta t + k(n + 1) \int_0^t \pi(r) dr}.
\]  

(4.3)

Note that the expression (4.2) confirms the nature of the boundary steady state \( P_1 \), in fact:

\[
\lim_{t \to +\infty} \pi(t) = \frac{1}{c}
\]  

(4.4)
for every choice of parameters and of \( u_0 \). It should be also noticed that the integration in (4.3) requires the use of special functions (see, for instance, Lebedev, 1972).

The game yields two steady state points:

\[
x_A^* = \frac{1}{2 \left[ c \beta + k (n-1) \right]} ; \quad u_A^* = \frac{1}{c} \\
x_B^* = \frac{(p - \beta - \rho) \left[ c(p + \beta + \rho) + 2k (n-1) \right]}{2 \left[ cp + k (n-1) \right] \left[ c \beta p + k (n-1) (p - \rho) \right]} ; \quad u_B^* = \frac{p - \beta - \rho}{pc + k (n-1)}
\]

(4.5)

where the first pair identifies the boundary solution, while the second one corresponds to the inner solution. If \( p > \beta + \rho \), both points have positive coordinates and therefore, at least in principle, are both admissible equilibria for all \( n \geq 2 \). The condition \( p > \beta + \rho \) says that the marginal value of an additional unit of output exceeds the sum of depreciation and discounting.

However, the economic feasibility of equilibria depends on profits. The associated steady state profits are:

\[
\pi_A^* = \frac{c(p - 2 \beta) - 2k (n-1)}{2c \left[ c \beta + k (n-1) \right]}
\]

(4.6)

\[
\pi_B^* = \frac{(p - \beta - \rho) \left[ cp - \beta + \rho \right] + 2k (n-1) \rho}{2 \left[ cp + k (n-1) \right] \left[ c \beta p + k (n-1) (p - \rho) \right]}
\]

(4.7)

From (4.6), we have that \( \pi_A^* \geq 0 \) for all \( n \leq 1 + c(p - 2 \beta) / (2k) \). This immediately entails that \( \beta > p/2 \) suffices to drive \( \pi_A^* \) below zero. Instead, from (4.7), we see that \( \pi_B^* \geq 0 \) for all \( p \geq \beta + \rho \) irrespective of the number of firms in the industry.

Accordingly, the admissibility of each equilibrium point listed in (4.5) can be assessed graphically in the space \( (p, n) \), as in Figure 1a,b. In both graphs, equilibrium \( B \) is not admissible in the half-plane where \( p < \beta + \rho \). Therefore, in such a range, we simply specify that \( u_B^* < 0 \).
Figure 1a: $\beta < \rho$

Figure 1b: $\beta > \rho$
These results, and their graphical illustration, deserve a comment. The inner solution depends on parameters \((p, \beta, \rho)\) but not on the industry structure (measured by the number of firms, \(n\)), while the admissibility of the boundary solution explicitly depends on the number of firms. The interpretation of the first condition is straightforward, as \(p > \beta + \rho\) means that the market is affluent enough to sustain the equilibrium with positive profits. As to the second condition, it says that an increase in the marginal revenue \(p\) allows for an increase in the maximum number of firms compatible with the boundary solution.

When both equilibria are admissible, one can derive a simple condition establishing the relative profitability of the two solutions, in the region where \(p > \max (2\beta, \beta + \rho)\) and \(n \leq \underline{n} = 1 + c \frac{(p - 2\beta)}{(2k)}\). Here, \(\pi_A^* > \pi_B^*\) for all \(n\) in the interval identified by

\[
n = \frac{(p - \rho) [4k - c (p + 2\beta)] \pm c \sqrt{p^2 (p - \rho)^2 + 4\beta (p^2 - \rho^2) (p - \beta)}}{4k (p - \rho)} \tag{4.8}
\]

with

\[
\begin{align*}
n_- &< 1 < n_+ < \underline{n} \quad \text{for all } \beta \in (0, \rho) \\
n_- &< n_+ < 1 < \underline{n} \quad \text{for all } \beta > \rho 
\end{align*}
\tag{4.9}
\]

Therefore, if \(\beta > \rho\) then \(\pi_A^* < \pi_B^*\) for all \(n \in (1, \underline{n})\); if instead \(\beta \in (0, \rho)\), we have \(\pi_A^* > \pi_B^*\) for all \(n \in [1, n_+]\) and \(\pi_A^* < \pi_B^*\) for all \(n \in (n_+, \underline{n})\).

5 Concluding remarks

We have extended the advertising game originally introduced by Leitmann and Schmitendorf (1978) to the case of \(n\) players, showing that (i) the open-loop Nash equilibrium remains strongly time consistent, (ii) the boundary solution is unaffected by the number of firms, while (iii) the inner solution does depend on industry structure. Then, focusing on the fully symmetric version of the game, we have identified the parameter constellations where both the inner and boundary solutions are feasible.
References


