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Fixed Agenda Social Choice Theory: Correspondence and Impossibility Theorems for Social Choice Correspondences and Social Decision Functions*

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Most impossibility results in the theory of social choice require that the collective choice rule operates on many different subsets of the universal set of alternatives. This paper shows that there is a close correspondence between this framework and a fixed agenda framework, i.e., one where it is only necessary to choose from one particular set of alternatives. It also shows that Arrow's Impossibility Theorem and Gibbard's oligarchy result can be translated into a fixed agenda framework. *Journal of Economic Literature* Classification Number: D70.

1. INTRODUCTION

Many contributions to the theory of social choice follow the *inter-agenda* approach as they assume that the collective choice rule can operate on many different agendas (i.e., many subsets of the universal set of all alternatives). Furthermore, the agenda domain is assumed to be sufficiently rich so as to give force to certain inter-agenda conditions, that is, axioms that require that if two agendas S and T meet certain conditions, then the choice sets out of S and T must bear some mutual relationships (e.g., collective rationality conditions and Independence of Irrelevant Alternatives). However, in actual situations, it is only necessary to choose from one particular subset of alternatives—the set of all alternatives which are feasible under the given circumstances. On the basis of this observation, it might be argued that Arrow's Impossibility Theorem and related impossibility results which rest heavily on *interagenda* conditions do not

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preclude the existence of a satisfactory collective choice rule.[1] Indeed, on a fixed agenda there exist efficient, anonymous, and neutral collective choice rules. The Borda counting rule is an easy example, but many other, more sophisticated methods can be devised.

However, new problems may arise if we impose *intra-agenda* conditions which are in the same spirit as the inter-agenda conditions of the Arrovian tradition. The main purpose of this paper is to show that there is a close correspondence between the fixed agenda and the multi-agenda frameworks: if a fixed agenda collective choice rule satisfies certain intra-agenda conditions, then there exists a Social Decision Function that satisfies appropriate inter-agenda conditions, and vice versa. As a consequence, some standard results which hold with a variable agenda have exact fixed agenda counterparts. In this paper, Arrow's Impossibility Theorem and Gibbard's oligarchy theorem for Quasi-transitive Social Decision Functions[2] are translated into a fixed agenda framework.

After some preliminaries in Section 2, we introduce in Section 3 two fixed agenda counterparts of the condition of Independence of Irrelevant Alternatives, i.e., Independence and Weak Independence. In Section 4 we show that, provided weak conditions of Pareto efficiency hold, the existence of a Social Choice Correspondence satisfying Independence (resp., Weak Independence) is equivalent to the existence of a Social Welfare Function (resp., a Quasi-transitive Social Decision Function) satisfying Independence of Irrelevant Alternatives. Exploiting this equivalence, we prove in Section 5 the fixed agenda counterparts of Arrow's Impossibility Theorem and of Gibbard's oligarchy result.

2. Notation and Definitions

Let $X = \{x, y, z, \ldots\}$ be the universal set of alternatives, with $\# X > 2$, and let $N = \{1, 2, \ldots, n\}$ be the finite set of individuals. Each individual i is endowed with a weak ordering R_i on X (i.e., a complete, reflexive and transitive binary relation on X). Strict preference P_i and indifference I_i are defined in the usual way.

Let W be the set of all weak orderings on X. Elements of the n-fold Cartesian product W^n are called preferences profiles and denoted by $p = \{R_1, R_2, \ldots, R_n\}$. Given a subset S of X, $p : S$ is the restriction of p to S.

[1] Bailey [1] has argued that Arrow's theorem rests in particular on the condition that social choice be defined on every pair of alternatives. For an assessment of Bailey's claim, cf. Donaldson and Weymark [6] and the literature cited therein.

[2] Though Gibbard is universally credited for this result, the first published proofs are due to Guha [8] and Mas Colell and Sonnenschein [10].

A Social Choice Correspondence (SCC) on X is a correspondence C which to any profile $p \in W^n$ on X assigns a non-empty subset of X, $C(p, X)$. $C(p, X)$ is the set of the choosable alternatives from X at profile p. Notice that we assume unrestricted domain.

Let R denote social preference, with P and I denoting strict social preference and social indifference, respectively. Recall that R is quasi-transitive if P is transitive. A Social Decision Function (SDF) on X is a function which to any profile $p \in W^n$ on X assigns an acyclic relation $R(p)$ on X. A Quasi-transitive Social Decision Function (Q-SDF) is an SDF the co-domain of which is restricted to the set of quasi-transitive preference relations on X; a Social Welfare Function (SWF) is an SDF the co-domain of which is further restricted to the set of weak orderings on X. Notice again that unrestricted domain is assumed throughout. If p, p', ... are distinct profiles, we denote by $R(p)$, $R(p')$, ... the corresponding social preference relations.

Next, we introduce two simple conditions which formalize the notion of weak Pareto efficiency. The first one is the well-known Weak Pareto Principle; the second one is the condition of Weak Pareto Optimality.

DEFINITION 1. An SDF satisfies the Weak Pareto Principle if, for all $x, y \in X$, $xP_i y$ for all $i \in N$ implies $xP(p)\, y$.

DEFINITION 2. An SCC satisfies Weak Pareto Optimality if, for all $x, y \in X$, $xP_i y$ for all $i \in N$ implies $y \notin C(p, X)$.

3. INDEPENDENCE OF IRRELEVANT ALTERNATIVES AND ITS FIXED AGENDA RELATIVES

The condition of Independence of Irrelevant Alternatives is based upon the notion that the relative desirability of any two alternatives is completely determined by information pertaining directly to that pair of alternatives. In the relation-theoretic framework, this notion is formalized as follows.

DEFINITION 3. An SDF satisfies the condition of Independence of Irrelevant Alternatives if, for all $x, y \in X$ and all $p, p' \in W^n$, if $p : \{x, y\} = p' : \{x, y\}$, then $xR(p)\, y$ if and only if $xR(p')\, y$.

Clearly, the relational version of the condition of Independence of Irrelevant Alternatives is restrictive only in a multi-agenda framework.[3] In order to retain in the fixed agenda framework the basic idea that only information pertaining directly to any two alternatives counts to determine their relative desirability we define social choice in terms of maximization of social preferences, and we consider the implications of the condition of (relational) Independence of Irrelevant Alternatives defined above.

Suppose to begin with that $R(p)$ is transitive. Suppose also that, at profile p, x is chosen and y is rejected. By the transitivity of R, this means that $xP(p)y$. Now consider a profile, p', whose restriction to $\{x, y\}$ is the same as that of p. By the condition of Independence of Irrelevant Alternatives, it must then be $xP(p')y$. It follows that y is dominated by x at profile p' and therefore cannot be chosen. We take this implication of the condition of Independence of Irrelevant Alternative as a primitive axiom.[4]

DEFINITION 4. An SCC satisfies the condition of Independence if, for all $x, y \in X$ and for all $p, p' \in W^n$, the following implication holds: if $x \in C(p, X)$, $y \notin C(p, X)$, and $p: \{x, y\} = p': \{x, y\}$, then $y \notin C(p', X)$.

A weaker conclusion is obtained if one postulates that $R(p)$ is quasi-transitive. Under quasi-transitivity, we cannot conclude that $xP(p)y$ from the fact that x is chosen and y is rejected at profile p: it may happen that y is dominated by x only indirectly, e.g., xIz, zPy, and xIy. However, if x were the *unique* alternative chosen at profile p, then we would be sure that $xP(p)y$ for any other alternative y, and we could conclude that y must still be dominated under profile p'. This motivates the following weakening of the condition of Independence.

[3] In the choice-theoretic framework a different version of the condition of Independence of Irrelevant Alternatives is commonly adopted. According to this formulation, $p : S = p' : S$ must imply $C(p, S) = C(p', S)$, for any admissible agenda S. But even this version of the independence condition runs into problems in a fixed agenda framework.

Consider a fixed agenda SCC. Then, if social choice is defined only on the universal set of alternatives X (as we assume in this paper), the choice-theoretic version of the condition of Independence of Irrelevant Alternatives does not impose any real restriction. To see why, notice that $p : X = p' : X$ means $p = p'$, so that $C(p, X) = C(p', X)$ follows by the very definition of an SCC.

If, more generally, the fixed agenda S is a proper subset of X, Independence of Irrelevant Alternatives does impose non-trivial restrictions. Still, if $\# S > 2$ it may happen that the relative desirability of two alternatives in S be affected by other alternatives which also belong to S. This makes it easy to find fixed agenda SCC's satisfying the standard Arrovian conditions: think for instance of the *restricted* Borda counting rule.

[4] Our Independence condition weakens a similar condition proposed by Hansson [9]. Hansson requires that for all $S \subseteq X$ and for all $p, p' \in W^n$, the following implication holds: if $S \cap C(p, X) \neq \emptyset$, $S \cap C(p', X) \neq \emptyset$, and $p : S = p' : S$, then $S \cap C(p, X) = S \cap C(p', X)$. It is easy to see that Hansson's condition implies Independence.

DEFINITION 5. An SCC satisfies the condition of Weak Independence if, for all $x, y \in X$ and for all $p, p' \in W^n$, the following implication holds: if $\{x\} = C(p, X)$ and $p: \{x, y\} = p': \{x, y\}$, then $y \notin C(p', X)$.

4. CORRESPONDENCE THEOREMS

In this section we shall show that there is a close correspondence between the fixed agenda and the multi-agenda frameworks. For sake of simplicity, the results are stated and proved for the case of a finite universal set of alternatives.

THEOREM 1. *Let X be finite. An SCC defined on X satisfying Weak Pareto Optimality and Independence exists iff there exists an SWF defined on X satisfying the Weak Pareto Principle and Independence of Irrelevant Alternatives.*[5]

Proof. (i) Sufficiency. Suppose there exists an SWF satisfying the Weak Pareto Principle and Independence of Irrelevant Alternatives. For all $p \in W^n$, let us define $C(p, X) = \{x; x \in X \text{ and } xR(p)\, y \text{ for all } y \in X\}$. Since X is finite and $R(p)$ is transitive for all p, $C(p, X)$ is always non-empty and therefore we have defined an SCC on X. We must now show that this SCC satisfies Weak Pareto Optimality and Independence. Weak Pareto Optimality follows immediately from the Weak Pareto Principle. We now prove that Independence holds. Consider two profiles, p and p', and two alternatives, x and y, such that $x \in C(p, X)$, $y \notin C(p, X)$, and $p: \{x, y\} = p': \{x, y\}$. We must prove that $y \notin C(p', X)$. We first show that $xP(p)\, y$. Indeed, since $x \in C(p, X)$, it must be $xR(p)z$ for all $z \in X$; hence, if $yR(p)x$, then by the transitivity of R it would follow $yR(p)z$ for all $z \in X$, and therefore $y \in C(p, X)$, violating our hypothesis. We conclude that $xP(p)\, y$. Now, from $xP(p)\, y$ and the condition of Independence of Irrelevant Alternatives if follows $xP(p')\, y$; hence $y \notin C(p', X)$.

(ii) Necessity. Suppose there exists an SCC satisfying Weak Pareto Optimality and Independence. We then define an SDF in the following way: for all $p \in W^n$ and for all $x, y \in X$, $xR(p)\, y$ iff $x \in C(p^*_{\{x,y\}}, X)$ and $xP(p)\, y$ iff $\{x\} = C(p^*_{\{x,y\}}, X)$, where $p^*_{\{x,y\}}$ is a profile obtained from p by shifting all alternatives different from x and y to the bottom of the preference ordering of each individual while leaving unaltered the relative

[5] Hansson [9] proves a similar theorem using a stronger independence condition (see also Fishburn [7]). Since Hansson's condition implies Independence (see footnote 4), our Theorems 1 and 3 imply his result.

ordering of x and y. More formally and generally, given a subset S of X and a profile p, we define p_S^* by the following conditions:

(a) $p : S = p_S^* : S$,

(b) for all $i \in N$, $x \in S$ and $y \in X - S$ imply $xP_i y$,

(c) for all $i \in N$, $y, x \in X - S$ implies $xI_i y$.

By Weak Pareto Optimality it follows that $R(p)$ is complete and reflexive. To prove transitivity, suppose that $xR(p)y$ and $yR(p)z$. This means that $x \in C(p^*_{\{x,y\}}, X)$ and $y \in C(p^*_{\{y,z\}}, X)$. We now show that it must be $x \in C(p^*_{\{x,y,z\}}, X)$. If not, by Weak Pareto Optimality only three cases can arise:

(i) $C(p^*_{\{x,y,z\}}, X) = \{y, z\}$. Since $x \in C(p^*_{\{x,y\}}, X)$, this violates Independence.

(ii) $C(p^*_{\{x,y,z\}}, X) = \{y\}$. Again, this violates Independence since $x \in C(p^*_{\{x,y\}}, X)$.

(iii) $C(p^*_{\{x,y,z\}}, X) = \{z\}$. Since $y \in C(p^*_{\{y,z\}}, X)$, this also violates Independence.

We have therefore $x \in C(p^*_{\{x,y,z\}}, X)$. This implies that $x \in C(p^*_{\{x,z\}}, X)$, for otherwise Independence would be violated. It follows $xR(p)z$, hence R is transitive.

It remains to be proven that the SWF we have defined satisfies the Weak Pareto Principle and Independence of Irrelevant Alternatives. The Weak Pareto Principle follows immediately from Weak Pareto Optimality. Independence of Irrelevant Alternatives follows from the observation that, if $p: \{x, y\} = p': \{x, y\}$, then $p^*_{\{x,y\}} = p'^*_{\{x,y\}}$ and therefore $C(p^*_{\{x,y\}}, X) = C(p'^*_{\{x,y\}}, X)$. Q.E.D.

THEOREM 2. *Let X be finite. An SCC defined on X satisfying Weak Pareto Optimality and Weak Independence exists iff there exists a Q-SDF defined on X satisfying the Weak Pareto Principle and Independence of Irrelevant Alternatives.*

Proof. (i) Sufficiency. Suppose there exists a Q-SDF satisfying the Weak Pareto Principle and Independence of Irrelevant Alternatives. For all $p \in W^n$, let us define $C(p, X) = \{x; x \in X$ and $xR(p)y$ for all $y \in X\}$. Since X is finite and $R(p)$ is quasi-transitive for all p, $C(p, X)$ is always non-empty and thus we have defined an SCC on X. Weak Pareto Optimality follows immediately from the Weak Pareto Principle. We now prove that Weak Independence holds. Consider two profiles, p and p', and two alternatives, x and y, such that $\{x\} = C(p, X)$ and $p: \{x, y\} = p': \{x, y\}$. We must prove that $y \notin C(p', X)$. We first show that $xP(p)y$. Indeed, since

$y \notin C(p, X)$, it must be $zP(p)\,y$ for some $z \in X$. If $z = x$ our claim is proved; if $z \neq x$, since z is not chosen, there must a w such that $wP(p)z$. Since X is finite and x is uniquely chosen, there must exist a finite sequence $xP(p) \cdots wP(p)\,zP(p)\,y$. By the quasi-transitivity of R it follows $xP(p)\,y$. Now, from $xP(p)\,y$ and the condition of Independence of Irrelevant Alternatives if follows $xP(p')\,y$; hence $y \notin C(p', X)$.

(ii) Necessity. Suppose there exists an SCC satisfying Weak Pareto Optimality and Weak Independence. On the basis of this SCC, we then define a Social Decision Function as in the proof of Theorem 1. To prove quasi-transitivity, suppose that $xP(p)\,y$ and $yP(p)z$. This means that $\{x\} = C(p^*_{\{x,y\}}, X)$ and $\{y\} = C(p^*_{\{y,z\}}, X)$. By Weak Independence, we have $y \notin C(p^*_{\{x,y,z\}}, X)$ and $z \notin C(p^*_{\{x,y,z\}}, X)$. By Weak Pareto Optimality, it follows $\{x\} = C(p^*_{\{x,y,z\}}, X)$. Hence, by Weak Independence we get $\{x\} = C(P^*_{\{x,z\}}, X)$. This implies $xP(p)z$.

The Weak Pareto Principle and Independence of Irrelevant Alternatives follow as in the proof of Theorem 1. Q.E.D.

5. Fixed Agenda Impossibility Theorems

The correspondence theorems proved in Section 4 allow us to obtain fixed agenda impossibility theorems from Arrow's and Gibbard's results. The strategy of proof is to show that, if an SCC satisfying Weak Pareto Optimality and Independence (resp., Weak Independence) is not dictatorial (resp., not oligarchic), then the corresponding SWF (resp., Q-SDF) is not dictatorial (resp., oligarchic), but this is not possible by the Arrow (resp., Gibbard) theorem.[6]

We first define the notions of dictatorial and oligarchic SDFs and SCCs. Let x and y be any two alternatives in X.

Definition 6. An SDF is: (i) *dictatorial* if there exists a $d \in N$ such that $xP_d\,y$ implies xPy; (ii) *oligarchic* if there exists a $L \subseteq N$ such that $xP_i\,y$ for all $i \in L$ implies xPy and, for all $j \in L$, $xP_j\,y$ implies xRy.

Definition 7. An SCC is: (i) *dictatorial* if there exists a $d \in N$ such that $xP_d\,y$ implies $y \notin C(p, X)$; (ii) *oligarchic* if there exists a $L \subseteq N$ such that

[6] It may be conjectured that the equivalence pointed out in this paper carries over to other conditions of collective rationality, so that, for instance, some further weakening of the Weak Independence condition would correspond to Independence of Irrelevant Alternatives plus acyclicity. If this conjecture were correct, then it should be possible to prove fixed agenda counterparts of the results for SDFs obtained, among others, by Blau and Deb [3] and by Blair and Pollak [2].

$xP_i y$ for all $i \in L$ implies $y \notin C(p, X)$ and, for all $j \in L$, $xP_j y$ implies $\{y\} \neq C(p, X)$.

We are now ready to prove the following theorems, which represent the fixed agenda versions of Arrow's Impossibility Theorems and of Gibbard's oligarchy result.

THEOREM 3. *Let X be finite. If an SCC defined on X satisfies Weak Pareto Optimality and Independence, then it is dictatorial.*

Proof. Consider an SCC which satisfies Weak Pareto Optimality and Independence, and define a SWF as in the proof of Theorem 1. Since, by Theorem 1, the SWF satisfies the Weak Pareto Principle and Independence of Irrelevant Alternatives, it must be dictatorial by Arrow's Impossibility Theorem. Let d be the dictator. Then, for all $x, y \in X$, $xP_d y$ implies xPy. But this means that $\{x\} = C(p^*_{\{x,y\}}, X)$, where p is any profile with $xP_d y$. By Independence it follows that y cannot be chosen whenever $xP_d y$, that is, d is a dictator for the original SCC. Q.E.D.

THEOREM 4. *Let X be finite. If an SCC defined on X satisfies Weak Pareto Optimality and Weak Independence, then it is oligarchic.*

Proof. Consider an SCC which satisfies Weak Pareto Optimality and Weak Independence, and define a Q-SDF as in the proof of Theorem 2. Since, by Theorem 2, the Q-SDF satisfies the Weak Pareto Principle and Independence of Irrelevant Alternatives, it must be oligarchic by Gibbard's theorem. Let L be the oligarchy. Then, for all $x, y \in X$, $xP_i y$ for all $i \in L$ implies xPy and $xP_j y$ for at least one $j \in L$ implies xRy. Consider first the case $xP_i y$ for all $i \in L$. Since xPy, it follows that $\{x\} = C(p^*_{\{x,y\}}, X)$, where p is any profile with $xP_i y$ for all $i \in L$. By Weak Independence, it follows that y cannot be chosen whenever $xP_i y$ for all $i \in L$, that is, L is a decisive coalition for the original SCC. Now suppose that $xP_j y$ for at least one $j \in L$. Since xRy, it follows that $x \in C(p^*_{\{x,y\}}, X)$, where p is any profile with $xP_j y$ for at least one $j \in L$. Now consider a profile p' with the same restriction to the pair $\{x, y\}$ as profile p. If $\{y\} = C(p', X)$, Weak Independence would be violated. It follows any individual in L has a veto power, which implies that L is an oligarchy. Q.E.D.

A direct proof of Theorems 3 and 4 may be found in Denicolò [4, 5].[7] Together with the correspondence Theorems 1 and 2, these results can be used to provide a new proof of Arrow's and Gibbard's theorems. Since the proofs are analogous to those of Theorems 3 and 4, they are left to the interested reader.

[7] Actually, the theorems proved in Denicolò [4, 5] are slightly more general, as they do not require the assumption that X be finite.

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