

# MONOPOLY WITH RESALE

## Online Appendix

GIACOMO CALZOLARI

ALESSANDRO PAVAN

October 2005

# 1 Restriction to price offers in the resale ultimatum bargaining game

In the model set up, we assume that in the resale ultimatum bargaining game  $B$  and  $T$  are restricted to simple take-it-or-leave-it price offers. In this appendix, we prove that, not only are price offers sequentially optimal for  $B$  and  $T$ , but  $S$  does not gain from recommending more complex mechanisms.<sup>1</sup> We first characterize the allocations (for the primary and the secondary market) that maximize the monopolist's revenue under minimal sequential rationality constraints for  $B$  and  $T$  and then show that these allocations can also be sustained in the game where offers in the resale bargaining game are restricted to simple prices.

For simplicity, assume  $\lambda_T = 1$  and consider the case where resale is to a third party (the proof for  $\lambda_T \in [0, 1]$  and inter-bidder resale follows similar arguments).

Suppose that at  $\tau = 2$ ,  $T$  can choose from a topological space of feasible resale mechanisms  $\Pi^r$ . A resale mechanism  $\pi = (\mathcal{M}^r, \alpha) \in \Pi^r$  consists of a set of messages  $\mathcal{M}^r$  for player  $B$  along with a measurable mapping  $\alpha : \mathcal{M}^r \rightarrow \mathbb{R} \times \Delta(\{0, 1\})$  that assigns to each message  $m^r \in \mathcal{M}^r$  a lottery over the decision to trade and an expected payment from  $T$  to  $B$ . Let  $\Upsilon$  denote the set of resale mechanisms that consist of simple take-it-or-leave-it price offers.<sup>2</sup> Without confusion, an element of  $\Upsilon$  can be denoted simply by the price  $t^r$ . Finally, let  $\Xi$  denote the set of direct resale mechanisms  $\xi : \Theta_B \mapsto \mathbb{R} \times \Delta(\{0, 1\})$ . Since any mechanism in  $\Xi$  has the same message space, to save on notation, an element of  $\Xi$  will be denoted simply by the mapping  $\xi$ .

Now, consider the monopolist. Let  $\Psi$  represent a topological space of feasible mechanisms for  $S$ . A mechanism  $\psi = (\mathcal{M}, \mathcal{R}, \beta) \in \Psi$  consists of a set of messages  $\mathcal{M}$  for  $B$ , a set of signals/recommendations  $\mathcal{R}$  that  $S$  can send to  $T$  and a measurable mapping  $\beta : \mathcal{M} \mapsto \mathbb{R} \times \Delta(\{0, 1\} \times \mathcal{R})$  that assigns to each message  $m \in \mathcal{M}$  an expected transfer  $t(m) \in \mathbb{R}$  from  $B$  to  $S$  and a joint lottery  $\delta(m) \in \Delta(\{0, 1\} \times \mathcal{R})$  over the decision to trade and the recommendations  $\mathcal{R}$ . Now, let  $\tilde{\Phi}$  denote the set of direct mechanisms  $\tilde{\phi} : \Theta_B \mapsto \mathbb{R} \times \Delta(\{0, 1\} \times \tilde{Z})$  in which the recommendations  $\tilde{z} = (\xi(\bar{\theta}_T), \xi(\underline{\theta}_T)) \in \tilde{Z} = \Xi^2$  that  $S$  sends to  $T$  consists in a pair of direct resale mechanisms, respectively for  $\bar{\theta}_T$  and  $\underline{\theta}_T$ . Finally, let  $\Phi$  denote the set of direct mechanisms  $\phi : \Theta_B \mapsto \mathbb{R} \times \Delta(\{0, 1\} \times Z)$  in which  $S$  recommends simple take-it-or-leave-it price offers  $z = (t^r(\bar{\theta}_T), t^r(\underline{\theta}_T)) \in Z = \Upsilon^2$ . In the following, we will denote by  $\tilde{\phi} \in \tilde{\Phi}$  an element of  $\tilde{\Phi}$  and by  $\phi \in \Phi$  an element of  $\Phi$ .

---

<sup>1</sup>This does not mean that a stochastic ultimatum bargaining game where  $B$  and  $T$  are randomly selected to design the resale mechanism is the most favorable resale procedure from the perspective of the initial seller. For example, in the case of inter-bidder resale, if  $S$  could choose the resale game, she could simply prohibit any future transaction between  $B$  and  $T$  and then implement a Myerson optimal auction in the primary market. Alternatively, she could dictate that it is always the resale-seller who makes the offer in the secondary market (as discussed in the paper, sometimes this also allows the monopolist to extract the Myerson revenue – see Zheng (2002)).

<sup>2</sup>Formally, a take-it-or-leave-it price offer is a mechanism  $(\mathcal{M}^r, \alpha)$ , with  $\mathcal{M}^r = \{yes, no\}$ , such that, when  $B$  chooses the message  $m = yes$ , the good is transferred to  $T$  and  $B$  receives a payment  $t^r$ , whereas when he chooses  $m = no$ , he keeps the good and receives no money from  $T$ .

Now, let  $U_S^*$  represent the highest equilibrium payoff for the monopolist in the restricted game where  $\Pi^r = \Upsilon$  and  $\Psi = \Phi$ , that is in the game where  $T$  can only make take-it-or-leave-it price offers and  $S$  can only offer direct mechanisms  $\phi \in \Phi$ , as assumed in the model set up. Similarly, let  $\tilde{U}_S^*$  denote the highest equilibrium payoff in an unrestricted game where  $\Pi^r \supseteq \Xi \cup \Upsilon$  and  $\Psi \supseteq \tilde{\Phi} \cup \Phi$ .

**Claim A1.** *S does not gain from recommending that T and B offer mechanisms in the ultimatum bargaining game more complex than simple price offers:  $\tilde{U}_S^* = U_S^*$ .*

**Proof.** We prove the result in three steps. Step 1 shows that in the unrestricted game,  $\tilde{U}_S^*$  can be sustained by an equilibrium where  $S$  offers a mechanism  $\tilde{\phi}^* \in \tilde{\Phi}$  and  $T$  follows the monopolist's recommendations. Step 2 characterizes the allocations induced by  $\tilde{\phi}^*$ . Finally, step 3 shows that these allocations can also be sustained by an equilibrium in the restricted game where  $\Pi^r = \Upsilon$  and  $\Psi = \Phi$ .

*Step 1.* Given any direct mechanism  $\tilde{\phi} \in \tilde{\Phi}$ , let  $\tilde{Z}(\tilde{\phi})$  denote the set of recommendations  $\tilde{z} = (\xi(\bar{\theta}_T), \xi(\underline{\theta}_T))$  in the support of  $\tilde{\phi}$  and  $\Xi(\tilde{\phi})$  the set of all direct resale mechanisms recommended by  $\tilde{\phi}$ . Formally,  $\Xi(\tilde{\phi}) = \{\xi \in \Xi : \exists \tilde{z} = (\xi(\bar{\theta}_T), \xi(\underline{\theta}_T)) \in \tilde{Z}(\tilde{\phi}) \text{ s.t. } \xi = \xi(\bar{\theta}_T) \text{ or } \xi = \xi(\underline{\theta}_T)\}$ .

Now, consider a mechanism  $\tilde{\phi}^*$  with the following properties:

- (i)  $B$  finds it optimal to participate and truthfully report his type in  $\tilde{\phi}^*$  as well as in any resale mechanism  $\xi \in \Xi(\tilde{\phi}^*)$ ;
- (ii) given any  $\tilde{z} = (\xi(\bar{\theta}_T), \xi(\underline{\theta}_T)) \in \tilde{Z}(\tilde{\phi}^*)$ , the direct mechanism  $\xi(\theta_T)$  is optimal for  $\theta_T$  – any other mechanism  $\xi \in \Xi$  that is individually-rational and incentive-compatible for  $B$  leads to a lower payoff for  $\theta_T$  (formally,  $\xi(\theta_T)$  is a solution to the program  $P_T(r, \theta_T)$  described below with  $r = \tilde{z}$ ).
- (iii)  $\tilde{\phi}^*$  is optimal for  $S$  – any other  $\tilde{\phi}$  that strictly dominates  $\tilde{\phi}^*$  necessarily violates (i) or (ii).

In the sequel, we prove the following two results. First, for any mechanism  $\tilde{\phi}^*$  that satisfies (i)-(iii), there exists an equilibrium in the unrestricted game where  $\Pi^r \supseteq \Xi \cup \Upsilon$  and  $\Psi \supseteq \tilde{\Phi} \cup \Phi$  that supports  $\tilde{\phi}^*$ . That is, we can construct a (sequentially rational) strategy for  $B$  that specifies a complete plan of action for any pair of mechanisms  $(\psi, \pi) \in \Psi \times \Pi^r$  – that is, a pair of function  $\sigma_B : \Theta_B \times \Psi \rightarrow \Delta(\mathcal{M})$  and  $\sigma_B : \Theta_B \times \Psi \times \mathcal{M} \times \mathbb{R} \times \{0, 1\} \times \mathcal{R} \times \Pi \rightarrow \Delta(\mathcal{M}^r)$  – and a (sequentially rational) strategy for  $T$  that specifies a reaction to any upstream mechanism  $\psi$  – that is, a function  $\sigma : \Theta_T \times \Psi \times \mathcal{R} \rightarrow \Delta(\Pi^r)$  – such that: (a)  $S$  finds it optimal to offer  $\tilde{\phi}^*$ ; (b)  $T$  finds it optimal to obey to the recommendations  $\tilde{z} \in \tilde{Z}(\tilde{\phi}^*)$ ; and (c)  $B$  finds it optimal to participate and truthfully report his type in  $\tilde{\phi}^*$  as well as in any resale mechanism  $\xi \in \Xi(\tilde{\phi}^*)$ . Second, the monopolist's payoff in the equilibrium supporting  $\tilde{\phi}^*$  is higher than in any other equilibrium, i.e. it yields  $\tilde{U}_S^*$ .<sup>3</sup>

To prove these claims, take any equilibrium of the unrestricted game. Given any upstream

---

<sup>3</sup>As we show below,  $\tilde{\phi}^*$  identifies a profile of allocations – probabilities of trade and transfers for each state  $(\theta_B, \theta_T)$  – that maximize the monopolist's payoff under minimal sequential rationality constraints for  $B$  and  $T$ .

mechanism  $\psi = (\mathcal{M}, \mathcal{R}, \beta) \in \Psi$ , let

$$\mathcal{R}(\sigma_B) := \{r \in \mathcal{R} : \exists m \in \text{Supp}[\sigma_B(\theta_B, \psi)] \text{ s.t. } r \in \text{Supp}[\delta(m)] \text{ for some } \theta_B \in \Theta_B\}$$

denote the set of recommendations that, given the buyer's strategy at  $\tau = 1$ , are sent with positive probability to  $T$ . For any recommendation  $r \in \mathcal{R}(\sigma_B)$ , the reaction  $\sigma_T(\theta_T, \psi, r) \in \Delta(\Pi^r)$  is sequentially rational for  $\theta_T$  if and only if, given the buyer's strategy at  $\tau = 2$ , it leads to a pair of probability of trade  $\{x^r(\bar{\theta}_B), x^r(\underline{\theta}_B)\} \in [0, 1]^2$  and a pair of expected transfers  $\{t^r(\bar{\theta}_B), t^r(\underline{\theta}_B)\} \in \mathbb{R}^2$  – that solve the following program:<sup>4</sup>

$$\mathcal{P}_T(r, \theta_T) : \begin{cases} \max_{x^r(\cdot), t^r(\cdot)} \sum_{\theta_B} [\theta_T x^r(\theta_B) - t^r(\theta_B)] \Pr(\theta_B | r; \psi) \\ \text{s.t. for any } (\theta_B, \hat{\theta}_B) \in \Theta_B^2 \\ t^r(\theta_B) - \theta_B x^r(\theta_B) \geq 0 \quad (IR_B(\theta_B)) \\ t^r(\theta_B) - \theta_B x^r(\theta_B) \geq t^r(\hat{\theta}_B) - \theta_B x^r(\hat{\theta}_B) \quad (IC_B(\theta_B)) \end{cases}$$

where  $\Pr(\theta_B | r; \psi)$  is computed using Bayes' rule and the buyer's strategy at  $\tau = 1$ . Indeed, if the allocations induced by  $\sigma_T(\theta_T, \psi, r)$  do not solve  $\mathcal{P}_T(r, \theta_T)$ , then,  $\theta_T$  has a profitable deviation that consists in offering a direct mechanism  $\xi \in \Xi$  which solves the above program.<sup>5</sup>

We conclude that, given any pair of (sequentially rational) strategies  $\sigma_B$  and  $\sigma_T$ , an upstream mechanism for the monopolist  $\psi = (\mathcal{M}, \mathcal{R}, \beta) \in \Psi$  (no matter whether it is on or off the equilibrium path) ultimately leads to a mapping  $f : \Theta_B \mapsto \mathbb{R} \times \Delta(\{0, 1\} \times \mathcal{R})$  that assigns to each  $\theta_B$  an expected transfer  $t(\theta_B) \in \mathbb{R}$  from  $B$  to  $S$  and a joint lottery  $\delta(\theta_B) \in \Delta(\{0, 1\} \times \mathcal{R})$ , with the following properties:

(A) type  $\theta_B \in \Theta_B$  prefers the outcome  $f(\theta_B) = (t(\theta_B), \delta(\theta_B))$  to the outcome  $f(\hat{\theta}_B) = (t(\hat{\theta}_B), \delta(\hat{\theta}_B))$  that can be obtained by mimicking the behavior of type  $\hat{\theta}_B \neq \theta_B$ .

(B) for any  $r \in \mathcal{R}(\sigma_B)$ , the allocations induced by the reaction of  $\theta_T$  solve  $\mathcal{P}_T(r, \theta_T)$ .

Consider the following transformation of  $\psi = (\mathcal{M}, \mathcal{R}, \beta)$  into a direct mechanism  $\tilde{\psi} \in \tilde{\Phi}$  (using  $\sigma_B$  and  $\sigma_T$ ). For any  $\tilde{z} = (\xi(\bar{\theta}_T), \xi(\underline{\theta}_T)) \in \tilde{Z} = \Xi^2$ , let  $\tilde{\mathcal{R}}(\tilde{z}) \subseteq \mathcal{R}(\sigma_B)$  denote the set of recommendations that, given  $\sigma_B$  and  $\sigma_T$ , ultimately lead to the same allocations as those specified in the pair of direct resale mechanisms  $(\xi(\bar{\theta}_T), \xi(\underline{\theta}_T))$ . Now, take a direct mechanism  $\tilde{\psi} : \Theta_B \mapsto \mathbb{R} \times \Delta(\{0, 1\} \times \tilde{Z})$  that assigns to each  $\theta_B \in \Theta_B$  the same expected transfer as  $\psi$ , and a lottery over  $\{0, 1\} \times \tilde{Z}$  such

<sup>4</sup>Note that, even if  $T$  faces an "informed principal" mechanism design problem, since both  $B$  and  $T$  have quasilinear preferences, private values and finite types,  $T$  never gains from hiding her private information to  $B$  – see Maskin and Tirole (1990).

<sup>5</sup>To guarantee that, whenever indifferent,  $B$  participates and truthfully reveals his type,  $T$  may need to increase the transfers  $t^r(\bar{\theta}_B)$  and  $t^r(\underline{\theta}_B)$ , that solve  $\mathcal{P}_T(r, \theta_T)$ , respectively by  $\varepsilon$  and  $\delta$ . However, with quasilinear payoffs,  $\varepsilon$  and  $\delta$  can be set arbitrarily close to zero.

that<sup>6</sup>

$$\tilde{\phi}(\tilde{z}|\theta_B) = \sum_{r \in \tilde{\mathcal{R}}(\tilde{z})} \psi(r|\theta_B)$$

where  $\psi(r|\theta_B) := \Pr(x = 1, r' = r|\theta_B; \psi)$  and  $\tilde{\phi}(\tilde{z}|\theta_B) := \Pr(x = 1, \tilde{z}' = \tilde{z}|\theta_B; \tilde{\phi})$ . The mechanism  $\tilde{\phi}$  constructed this way maps  $\Theta_B$  into the same final outcomes – probability of trade and expected payments – as the mechanism  $\psi$ . Furthermore, given  $\tilde{\phi}$ ,  $T$  has the correct incentives to follow the monopolist's recommendations. To see this, note that when the supports  $\Theta_B$  and  $\Theta_T$  overlap, a recommendation  $\tilde{z} = (\xi(\bar{\theta}_T), \xi(\underline{\theta}_T)) \in \tilde{Z}(\tilde{\phi})$ , is incentive-compatible if and only if  $\xi(\underline{\theta}_T)$  is such that

$$x^r(\underline{\theta}_B) = 1, \quad x^r(\bar{\theta}_B) = 0, \quad t^r(\underline{\theta}_B) = \underline{\theta}_B, \quad t^r(\bar{\theta}_B) = 0, \quad (SR(\tilde{z}, \underline{\theta}_T))$$

and  $\xi(\bar{\theta}_T)$  is such that

$$\begin{aligned} x^r(\underline{\theta}_B) = 1; \quad x^r(\bar{\theta}_B) = & \begin{cases} 1 & \text{if } \Pr(\bar{\theta}_B|\tilde{z}) > \Delta\theta_B/[\bar{\theta}_T - \underline{\theta}_B] \\ 0 & \text{if } \Pr(\bar{\theta}_B|\tilde{z}) < \Delta\theta_B/[\bar{\theta}_T - \underline{\theta}_B] \\ \text{any } \eta \in [0, 1] & \text{if } \Pr(\bar{\theta}_B|\tilde{z}) = \Delta\theta_B/[\bar{\theta}_T - \underline{\theta}_B] \end{cases} \\ t^r(\bar{\theta}_B) = x^r(\bar{\theta}_B)\bar{\theta}_B; \quad t^r(\underline{\theta}_B) = & \underline{\theta}_B + x^r(\bar{\theta}_B)\Delta\theta_B \end{aligned} \quad (SR(\tilde{z}, \bar{\theta}_T))$$

Hence, recommendations may differ only with respect to what  $S$  recommends to  $\bar{\theta}_T$ . Now, take a recommendation  $\tilde{z} \in \tilde{Z}(\tilde{\phi})$  such that  $\xi(\bar{\theta}_T) = (x^r(\bar{\theta}_B) = x^r(\underline{\theta}_B) = 1; t^r(\bar{\theta}_B) = t^r(\underline{\theta}_B) = \bar{\theta}_B)$ . Then, for any  $r \in \tilde{\mathcal{R}}(\tilde{z})$ ,

$$\Pr(\bar{\theta}_B|r; \psi) = \frac{\psi(r|\theta_B)\Pr(\bar{\theta}_B)}{\psi(r|\theta_B)\Pr(\theta_B) + \psi(r|\underline{\theta}_B)\Pr(\underline{\theta}_B)} \geq \Delta\theta_B/[\bar{\theta}_T - \underline{\theta}_B].$$

Since, given  $\tilde{\phi}$ ,  $T$ 's posterior beliefs when she receives the recommendation  $\tilde{z}$  are given by

$$\Pr(\bar{\theta}_B|\tilde{z}; \tilde{\phi}) = \frac{\tilde{\phi}(\tilde{z}|\bar{\theta}_B)\Pr(\bar{\theta}_B)}{\tilde{\phi}(\tilde{z}|\bar{\theta}_B)\Pr(\bar{\theta}_B) + \tilde{\phi}(\tilde{z}|\underline{\theta}_B)\Pr(\underline{\theta}_B)} = \frac{\sum_{r \in \tilde{\mathcal{R}}(\tilde{z})} \psi(r|\bar{\theta}_B)\Pr(\bar{\theta}_B)}{\sum_{r \in \tilde{\mathcal{R}}(\tilde{z})} \psi(r|\bar{\theta}_B)\Pr(\bar{\theta}_B) + \sum_{r \in \tilde{\mathcal{R}}(\tilde{z})} \psi(r|\underline{\theta}_B)\Pr(\underline{\theta}_B)},$$

then  $\Pr(\bar{\theta}_B|\tilde{z}; \tilde{\phi}) \geq \Delta\theta_B/[\bar{\theta}_T - \underline{\theta}_B]$ , which implies that  $\tilde{z}$  is indeed incentive-compatible. The same result can be established for any  $\tilde{z} \in \tilde{Z}(\tilde{\phi})$ .

We conclude that for any mechanism  $\psi$ , there exists a mechanism  $\tilde{\phi}$  satisfying (i) and (ii) that is payoff-equivalent for all players. >From (iii), it is then immediate that in the unrestricted game, there exists an equilibrium sustaining  $\tilde{\phi}^*$ . Furthermore, the monopolist's payoff in such an equilibrium is necessarily (weakly) higher than in any other equilibrium of the unrestricted game.

*Step 2.* Now, let  $r(\theta_B; \xi) := t^r(\theta_B) - \xi(1|\theta_B)\theta_B$  denote the resale surplus that  $\theta_B$  obtains when  $T$  offers a direct resale mechanism  $\xi$ , and  $r(\theta_B|\tilde{z}) := p_T r(\theta_B; \xi(\bar{\theta}_T)) + (1 - p_T)r(\theta_B; \xi(\underline{\theta}_T))$  the

<sup>6</sup>For simplicity, we assume  $\mathcal{R}(\tilde{z})$  is a finite set. If not, then let  $\tilde{\phi}(\tilde{z}|\theta_B) = \int_{r \in \mathcal{R}(\tilde{z})} d\delta(r|\theta_B)$ , where  $\delta(r|\theta_B)$  denotes the probability measure of recommendation  $r$  induced by the buyer's strategy at  $\tau = 1$ .

expected surplus given the recommendation  $\tilde{z} = (\xi(\bar{\theta}_T), \xi(\underline{\theta}_T))$ . The mechanism  $\tilde{\phi}^*$  satisfies (i)-(iii) if and only if it is a solution to the following program

$$\tilde{\mathcal{P}}_S : \begin{cases} \max_{\tilde{\phi} \in \tilde{\Phi}} \mathbb{E}_{\theta_B} [t(\theta_B)] \\ \text{s.t.} - \text{ for any } (\theta_B, \hat{\theta}_B) \in \Theta_B^2 - \\ U(\theta_B) := \sum_{\tilde{z} \in \tilde{Z}} \tilde{\phi}(\tilde{z}|\theta_B) \{\theta_B + r(\theta_B|\tilde{z})\} - t(\theta_B) \geq 0 \\ U(\theta_B) \geq \sum_{\tilde{z} \in \tilde{Z}} \tilde{\phi}(\tilde{z}|\hat{\theta}_B) \{\theta_B + r(\theta_B|\tilde{z})\} - t(\hat{\theta}_B) \\ \text{for any } \tilde{z} \in \tilde{Z}(\tilde{\phi}) \text{ and any } \theta_T \in \Theta_T, \xi(\theta_T) \text{ satisfies } SR(\tilde{z}, \theta_T) \\ \tilde{\phi}(\tilde{z}|\theta_B) \geq 0 \text{ with } \sum_{z \in Z} \tilde{\phi}(\tilde{z}|\theta_B) \leq 1 \text{ for any } \theta_B \in \Theta_B \quad (\mathcal{F}) \end{cases}$$

*Step 3.* Note that  $S$  never gains from using a mechanism  $\tilde{\phi}$  that recommends a  $\xi(\bar{\theta}_T)$  in which  $x^r(\bar{\theta}_B) \in (0, 1)$ . Indeed, for any such mechanism, there exists another mechanism  $\tilde{\phi}'$  in which  $S$  recommends only  $\xi(\bar{\theta}_T)$  such that either  $x^r(\bar{\theta}_B) = 0$ , or  $x^r(\bar{\theta}_B) = 1$ , which leads to a higher payoff. It follows that  $S$  sends only two possible incentive-compatible recommendations: the first one is for both  $\bar{\theta}_T$  and  $\underline{\theta}_T$  to trade only with  $\underline{\theta}_B$  at a price  $t^r = \underline{\theta}_B$ ; the second is for  $\bar{\theta}_T$  to trade only with  $\underline{\theta}_B$  at a price  $t^r = \underline{\theta}_B$  and for  $\bar{\theta}_T$  to trade with both types at a price  $t^r = \bar{\theta}_B$ . But these are exactly the same resale outcomes that can be implemented recommending simple take-it-or-leave-it price offers. It is then immediate that the solution to  $\tilde{\mathcal{P}}_S$  leads exactly to the same revenue as the solution to  $\mathcal{P}_S$  in the main text. We conclude that  $\tilde{U}_S^*$  can also be achieved in the game where  $\Pi^r = \Upsilon$  and  $\Psi = \Phi$ . ■

## 2 Implementation of the optimal mechanism of Proposition 1 with price disclosures

**Claim A2.** *When the direct mechanism of Proposition 1 can not be implemented announcing only the decision to trade, it suffices to disclose the price to implement the optimal informational linkage with the secondary market.*

**Proof.** The implementations in which  $S$  discloses the price but keeps the choice of the contract secret, or discloses the contract with probability less than one, are immediate. In what follows, we prove that  $S$  could also fully disclose the choice of the contract by inducing  $B$  to play a mixed strategy.

Suppose  $S$  offers a menu of two price-lottery pairs. The menu is such that  $B$  receives the good with certainty if he pays  $t_H = t^*(\bar{\theta}_B)$  and with probability  $\delta = [1 - J/K]/[1 - J]$  if he pays  $t_L = \delta [\underline{\theta}_B + \lambda_B s(\underline{\theta}_B)]$ , where  $t^*(\bar{\theta}_B)$  is the price  $\bar{\theta}_B$  pays in the direct mechanism of Proposition 1.

We want to show that it is an equilibrium for the high type to pay  $t_H$  and for the low type to randomize over  $t_H$  and  $t_L$  with probability respectively equal to  $J$  and  $1 - J$ . Given this strategy,  $\bar{\theta}_T$  offers  $t^r = \bar{\theta}_B$  when she observes  $t_H$  and  $t^r = \underline{\theta}_B$  when she observes  $t_L$ , that is  $t_H$  and  $t_L$  serve the same role as  $\bar{z}$  and  $\underline{z}$  in the direct mechanism. For the low type to be indifferent between  $t_H$  and  $t_L$  it must be that

$$\underline{\theta}_B + \lambda_B s(\underline{\theta}_B) + \lambda_T p_T \Delta \theta_B - t_H = \delta [\underline{\theta}_B + \lambda_B s(\underline{\theta}_B)] - t_L. \quad (1)$$

Since  $t_H = t^*(\bar{\theta}_B)$ , the left hand side in (1) is also equal to the payoff  $\underline{\theta}_B$  obtains by announcing  $\theta_B = \bar{\theta}_B$  in the direct mechanism, which is equal to zero since  $IC(\underline{\theta}_B)$  and  $IR(\underline{\theta}_B)$  bind in the optimal mechanism. As a consequence,  $t_L = \delta [\underline{\theta}_B + \lambda_B s(\underline{\theta}_B)]$ .

Next, we prove that the high type is also indifferent between  $t_H$  and  $t_L$ , that is  $\bar{\theta}_B - t_H = \delta [\bar{\theta}_B + \lambda_B s(\bar{\theta}_B)] - t_L$ . Using the values of  $\delta$  and  $t_L$ , the previous equality is equivalent to

$$\bar{\theta}_B - t_H - [\Delta \theta_B + \lambda_B \Delta s - \lambda_T p_T \Delta \theta_B] = 0$$

which holds true since  $t_H = t^*(\bar{\theta}_B)$  and in the optimal mechanism both  $IR(\underline{\theta}_B)$  and  $IC(\bar{\theta}_B)$  are binding, which implies that  $0 = \bar{\theta}_B - t_H - [\Delta \theta_B + \lambda_B \Delta s - \lambda_T p_T \Delta \theta_B]$ .

Since this mechanism gives  $B$  the same payoff and induces the same distribution over  $x$  and  $Z$  as the optimal direct mechanism, it must also give  $S$  the same expected revenue. ■

### 3 Resale to third parties with multiple bidders in the primary market

**Claim A3.** *A monopolist always benefits from the existence of a secondary market when she is not able to contract with all potential buyers and resale can only be to a third party who does not participate in the primary market.*

**Proof.** Assume there are  $N \geq 2$  potential buyers in the primary market. At the end of the auction, the winner may keep the good for himself or resell it to  $T$  in the secondary market, in which case the bargaining game is exactly as in the single-bidder case with  $\lambda_i$  denoting the relative bargaining power of bidder  $i$  with respect to  $T$ . Continue to assume A1-A4 hold for each bidder and let  $\theta_B := (\theta_1, \theta_2, \dots, \theta_N) \in \Theta_B := \prod_{i=1}^N \Theta_i$  denote a profile of independent private values. Following the same steps as for the single bidder case, one can show that, conditional on bidder  $i$  winning the auction,  $S$  needs to send only two recommendations:  $\underline{z}^i$  must induce  $\bar{\theta}_T$  to offer  $t^r(\bar{\theta}_T) = \underline{\theta}_i$  and  $\bar{z}^i$  to offer  $t^r(\bar{\theta}_T) = \bar{\theta}_i$ . Let  $\phi(z^i | \theta_B)$  denote the probability the good is assigned to bidder  $i$  and a recommendation  $z^i \in \{\bar{z}^i, \underline{z}^i\}$  is sent to  $T$  when the bidders report  $\theta_B$ . Also, let

$$V(\theta_i | z^i) := \bar{\theta}_i + \lambda_i s_i(\bar{\theta}_i)$$

$$V_i(\underline{\theta}_i|z^i) := \underline{\theta}_i - \frac{p_i}{1-p_i}\Delta\theta_i + \lambda_i\{s_i(\underline{\theta}_i) - \frac{p_i}{1-p_i}\Delta s_i\} + (1-\lambda_i)\{r_i(\underline{\theta}_i|z^i) - \frac{p_i}{1-p_i}\Delta r_i(z^i)\}$$

denote the resale-augmented virtual valuations of bidder  $i$ . Following the same steps as for the single bidder case, we can show that an optimal auction  $\phi^*$  maximizes

$$\mathbb{E}_{\theta_B} \left[ \sum_{i=1}^N \sum_{z^i \in \{\bar{z}^i, \underline{z}^i\}} V(\theta_i|z^i) \phi(z^i|\theta_B) \right]$$

subject to

$$\mathbb{E}_{\theta_{-i}} \left\{ \sum_{z^i \in \{\bar{z}^i, \underline{z}^i\}} \phi(z^i|\bar{\theta}_i, \theta_{-i}) [\Delta\theta_i + \lambda_i\Delta s_i + (1-\lambda_i)\Delta r_i(z^i)] \right\} \geq \mathbb{E}_{\theta_{-i}} \left\{ \sum_{z^i \in \{\bar{z}^i, \underline{z}^i\}} \phi(z^i|\underline{\theta}_i, \theta_{-i}) [\Delta\theta_i + \lambda_i\Delta s_i + (1-\lambda_i)\Delta r_i(z^i)] \right\} \quad (IC(\underline{\theta}_i))$$

$$\Pr(\bar{\theta}_i|\underline{z}^i) \leq \frac{\Delta\theta_i}{\bar{\theta}_T - \underline{\theta}_i}, \quad (IC(\underline{z}^i))$$

$$\Pr(\bar{\theta}_i|\bar{z}^i) \geq \frac{\Delta\theta_i}{\bar{\theta}_T - \underline{\theta}_i}, \quad (IC(\bar{z}^i))$$

for  $i \in \{1, 2, \dots, N\}$  and  $\theta_{-i} := (\theta_1, \theta_2, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_N)$ .

To prove the claim, we compare the expected revenue associated with the solution to the above program with the revenue  $S$  could achieve in a Myerson optimal auction without resale. Recall that for any type profile  $\theta_B$ , a Myerson auction consists in assigning the good to the bidder with the highest virtual valuation,  $M(\theta_i)$ , provided that  $\max_i \{M(\theta_i)\} \geq 0$ , and in withholding the good otherwise. The expected revenue of a Myerson optimal auction is thus  $\mathbb{E}_{\theta_B} [\max\{0, M(\theta_1), \dots, M(\theta_N)\}]$ , where  $M(\bar{\theta}_i) := \bar{\theta}_i$  and  $M(\underline{\theta}_i) := \underline{\theta}_i - \frac{p_i}{1-p_i}\Delta\theta_i$ , for each  $i = 1, \dots, N$ .

The proof is in two steps. The first step proves that for any  $\theta_i \in \Theta_i$  and  $z^i$ , the resale-augmented virtual valuations are higher than the corresponding Myerson virtual valuations; that is,  $V(\theta_i|z^i) \geq M(\theta_i)$ . This follows directly from the fact that  $s(\theta_i) \geq 0$ ,  $r_i(z^i) \geq 0$ ,  $\Delta s_i \leq 0$  and  $\Delta r_i(z^i) \leq 0$ , for any  $z^i \in \{\bar{z}^i, \underline{z}^i\}$  and any  $i$ .

The second step proves that there exists a recommendation policy that allows to implement Myerson allocation rule with resale. Conditional on  $i$  winning the auction, suppose  $S$  sends only one recommendation  $z^i \in \{\bar{z}^i, \underline{z}^i\}$ , independently of whether  $i$  announces a low or a high type. The particular recommendation  $S$  sends to  $T$  depends on the posterior beliefs that are generated by the Myerson allocation rule; that is,  $S$  recommends  $z^i = \underline{z}^i$  if  $\Pr(\bar{\theta}_i|i) \leq \frac{\Delta\theta_i}{\bar{\theta}_T - \underline{\theta}_i}$ , and  $z^i = \bar{z}^i$  otherwise, where  $\Pr(\bar{\theta}_i|i)$  denotes the probability that  $\theta_i = \bar{\theta}_i$  given that bidder  $i$  wins the auction. Given this policy, which can be trivially implemented disclosing only the identity of the winner,  $IC(\underline{z}^i) - IC(\bar{z}^i)$  are clearly satisfied. Furthermore, since Myerson allocation rule is monotonic – i.e.  $\mathbb{E}_{\theta_{-i}} \{\phi(z^i|\bar{\theta}_i, \theta_{-i})\} \geq \mathbb{E}_{\theta_{-i}} \{\phi(z^i|\underline{\theta}_i, \theta_{-i})\}$ , constraints  $IC(\underline{\theta}_i)$  are also satisfied for each  $i$ . It follows that Myerson allocation rule remains implementable also in the presence of resale. It is then immediate that the optimal mechanism  $\phi^*$  must satisfy

$$\mathbb{E}_{\theta_B} \left[ \sum_{i=1}^N \sum_{z^i \in \{\bar{z}^i, \underline{z}^i\}} V(\theta_i|z^i) \phi^*(z^i|\theta_B) \right] \geq \mathbb{E}_{\theta_B} [\max\{0, M(\theta_1), \dots, M(\theta_N)\}],$$

which proves the result. ■

## 4 Resale to third parties: collusion in the primary market

When  $S$  lacks of the commitment not to collude with  $B$ , the only credible information that can be disclosed to the secondary market is the decision to trade. Furthermore, the possibility for  $S$  to make  $\phi$  public has no strategic effect so that  $\phi$  must be a best response to the strategy  $T$  is expected to follow in the secondary market. The optimal mechanism can be designed by looking at the value of the (collusion proof) resale-augmented virtual valuations

$$\begin{aligned} V(\bar{\theta}_B|\gamma) &:= \bar{\theta}_B + \lambda_B s(\bar{\theta}_B), \\ V(\underline{\theta}_B|\gamma) &:= \underline{\theta}_B - \frac{p_B}{1-p_B} \Delta\theta_B + \lambda_B \left[ s(\underline{\theta}_B) - \frac{p_B}{1-p_B} \Delta s \right] + \lambda_T \gamma \left[ \Delta\theta_B + \frac{p_B}{1-p_B} \Delta\theta_B \right], \end{aligned}$$

where  $\gamma \in [0, p_T]$  is the probability  $T$  is expected to offer a high price in the resale game. The seller's optimal (collusion-proof) mechanism then maximizes  $U_S := \mathbb{E}_{\theta_B} [V(\theta_B|\gamma)\phi(\theta_B)]$  under the monotonicity condition  $\phi(\bar{\theta}_B) \geq \phi(\underline{\theta}_B)$ , where  $\phi(\theta_B)$  denotes the probability of trade when  $B$  reports  $\theta_B$ .

To see how the informational linkage with the secondary market can be fashioned through a stochastic allocation rule, assume  $T$ 's prior beliefs are unfavorable ( $J < 1$ ) and  $V(\underline{\theta}_B|0) < 0 < V(\underline{\theta}_B|p_T)$ . In the unique equilibrium,  $S$  sells to  $\underline{\theta}_B$  with probability  $J$  and to  $\bar{\theta}_B$  with certainty.  $\bar{\theta}_T$  is then indifferent between offering a high and a low price and randomizes offering  $t^r(\bar{\theta}_T) = \bar{\theta}_B$  with probability  $\gamma^* \in (0, 1)$  and  $t^r(\bar{\theta}_T) = \underline{\theta}_B$  with probability  $1 - \gamma^*$ , where  $\gamma^*$  solves  $V(\underline{\theta}_B|p_T\gamma^*) = 0$  and hence makes  $S$  indifferent between selling to the low type and retaining the good.

## 5 Extended proof of Proposition 2: Optimal auctions with inter-bidder resale

The reduced program is in the Appendix of the paper (proof of Proposition 2). Here we derive a complete characterization of the optimal mechanism in the two polar cases where  $\lambda_T = 1$  and  $\lambda_B = 1$ .

**$T$  has all bargaining power (i.e.  $\lambda_T = 1$ ).**

In this case,  $S$  does not need to disclose any information to  $B$ . Therefore, we eliminate  $z_B$  from the mechanism  $\phi$ . Furthermore, since  $\underline{\theta}_B \leq \underline{\theta}_T \leq \bar{\theta}_B \leq \bar{\theta}_T$ , when  $h = T$ , the only incentive-compatible recommendation for  $\bar{\theta}_T$  is to ask  $t^r(\bar{\theta}_T) \geq \bar{\theta}_T$  and for  $\underline{\theta}_T$  to ask  $t^r(\underline{\theta}_T) = \bar{\theta}_B$  if  $\Pr(\bar{\theta}_B|\cdot) > 0$  and any  $t^r \geq \underline{\theta}_T$  otherwise. Similarly, when  $h = B$ , the only incentive-compatible recommendation

for  $\underline{\theta}_T$  is to offer a price  $t^r(\underline{\theta}_T) = \underline{\theta}_B$  if  $\Pr(\underline{\theta}_B|\cdot) > 0$  and any price  $t^r(\underline{\theta}_T) \leq \underline{\theta}_T$  otherwise. Without loss, we will assume that  $\underline{\theta}_T$  always asks  $t^r(\underline{\theta}_T) = \bar{\theta}_B$  when  $h = T$  and offers  $t^r(\underline{\theta}_T) = \underline{\theta}_B$  when  $h = B$ .<sup>7</sup> Since these are the same prices that  $T$  offers in the absence of any explicit recommendation, to save on notation, we will drop  $z_B$  from the mapping  $\phi$  when  $h = T$ , or  $h = B$  and  $\theta_T = \underline{\theta}_T$ . When instead  $h = B$  and  $\theta_T = \bar{\theta}_T$ , we will denote by  $\bar{z}$  and  $\underline{z}$  the recommendations (for  $\bar{\theta}_T$ ) to offer  $t^r(\bar{\theta}_T) = \bar{\theta}_B$  and  $t^r(\bar{\theta}_T) = \underline{\theta}_B$ , respectively. For these recommendations to be incentive compatible, the mechanism  $\phi$  must satisfy

$$\begin{aligned} \phi(B, \underline{z}|\bar{\theta}_T, \underline{\theta}_B) &\geq J\phi(B, \underline{z}|\bar{\theta}_T, \bar{\theta}_B) & \widetilde{IC}(\underline{z}, \bar{\theta}_T) \\ \phi(B, \bar{z}|\bar{\theta}_T, \underline{\theta}_B) &\leq J\phi(B, \bar{z}|\bar{\theta}_T, \bar{\theta}_B) & \widetilde{IC}(\bar{z}, \bar{\theta}_T) \end{aligned}$$

where  $J := \frac{p_B(\bar{\theta}_T - \bar{\theta}_B)}{(1-p_B)\Delta\theta_B}$ . The maximal revenue for the monopolist can be derived by partitioning the set of direct mechanisms  $\Phi$  into two classes. The first one, which we denote by  $\Phi_1$ , is such that  $\bar{\theta}_T$  offers a price  $t^r(\bar{\theta}_T) = \bar{\theta}_B$  off-equilibrium, after reporting  $\theta_T = \underline{\theta}_T$ . The second,  $\Phi_2$ , is such that  $\bar{\theta}_T$  offers  $t^r(\bar{\theta}_T) = \underline{\theta}_B$ . For a mechanism  $\phi$  to belong to  $\Phi_1$  it must be that

$$\phi(B|\underline{\theta}_T, \underline{\theta}_B) \leq J\phi(B|\underline{\theta}_T, \bar{\theta}_B) \quad C_1$$

Similarly, if  $\phi \in \Phi_2$ , then

$$\phi(B|\underline{\theta}_T, \underline{\theta}_B) \geq J\phi(B|\underline{\theta}_T, \bar{\theta}_B) \quad C_2$$

Letting  $\mathbb{I}(\phi) = 1$  if  $\phi \in \Phi_1$ ,  $\mathbb{I}(\phi) = 0$  if  $\phi \in \Phi_2$ , and substituting the values of  $s_T(\cdot)$  and  $r_B(\cdot)$ , the problem for the monopolist reduces to the choice of a mechanism  $\phi^*$  that maximizes

$$\begin{aligned} U_S &= p_T p_B \{ \bar{\theta}_T \phi(T|\bar{\theta}_T, \bar{\theta}_B) + \bar{\theta}_T \phi(B, \bar{z}|\bar{\theta}_T, \bar{\theta}_B) + \bar{\theta}_B \phi(B, \underline{z}|\bar{\theta}_T, \bar{\theta}_B) \} \\ &+ p_T (1-p_B) \{ \bar{\theta}_T \phi(T|\bar{\theta}_T, \underline{\theta}_B) + \bar{\theta}_T \phi(B, \bar{z}|\bar{\theta}_T, \underline{\theta}_B) + (\bar{\theta}_T - \frac{p_B}{1-p_B} \Delta\theta_B) \phi(B, \underline{z}|\bar{\theta}_T, \underline{\theta}_B) \} \\ &+ (1-p_T) p_B \{ [\bar{\theta}_B - \frac{p_T}{1-p_T} (\bar{\theta}_T - \bar{\theta}_B)] [\phi(T|\underline{\theta}_T, \bar{\theta}_B) + \mathbb{I}(\phi) \phi(B|\underline{\theta}_T, \bar{\theta}_B)] + \bar{\theta}_B [1 - \mathbb{I}(\phi)] \phi(B|\underline{\theta}_T, \bar{\theta}_B) \} \\ &+ (1-p_T) (1-p_B) \{ M(\underline{\theta}_T) \phi(T|\underline{\theta}_T, \underline{\theta}_B) + \mathbb{I}(\phi) [M(\underline{\theta}_T) + (\frac{p_T}{1-p_T} - \frac{p_B}{1-p_B}) \Delta\theta_B] \phi(B|\underline{\theta}_T, \underline{\theta}_B) \\ &+ [1 - \mathbb{I}(\phi)] (M(\underline{\theta}_T) - \frac{p_B}{1-p_B} \Delta\theta_B) \phi(B|\underline{\theta}_T, \underline{\theta}_B) \} \end{aligned}$$

subject to  $\widetilde{IC}(\bar{z}, \bar{\theta}_T)$ ,  $\widetilde{IC}(\underline{z}, \bar{\theta}_T)$ ,  $C_1$ ,  $C_2$ , and

$$\begin{aligned} &p_T [\phi(B, \underline{z}|\bar{\theta}_T, \bar{\theta}_B) - \phi(B, \underline{z}|\bar{\theta}_T, \underline{\theta}_B)] + (1-p_T) \mathbb{I}(\phi) [\phi(B|\underline{\theta}_T, \bar{\theta}_B) - \phi(B|\underline{\theta}_T, \underline{\theta}_B)] \\ &+ (1-p_T) [1 - \mathbb{I}(\phi)] [\phi(B|\underline{\theta}_T, \bar{\theta}_B) - \phi(B|\underline{\theta}_T, \underline{\theta}_B)] \geq 0 \quad (\widetilde{IC}(\underline{\theta}_B)) \end{aligned}$$

<sup>7</sup>Clearly,  $S$  does not have any incentive to recommend a different price.

$$\begin{aligned}
& p_B (\bar{\theta}_T - \bar{\theta}_B) [\phi(T|\bar{\theta}_T, \bar{\theta}_B) + \phi(B, \bar{z}|\bar{\theta}_T, \bar{\theta}_B) - \phi(T|\underline{\theta}_T, \bar{\theta}_B) - \mathbb{I}(\phi)\phi(B|\underline{\theta}_T, \bar{\theta}_B)] \\
& + (1 - p_B) \Delta\theta_T [\phi(T|\bar{\theta}_T, \underline{\theta}_B) + \phi(B, \bar{z}|\bar{\theta}_T, \underline{\theta}_B) - \phi(T|\underline{\theta}_T, \underline{\theta}_B) - [1 - \mathbb{I}(\phi)]\phi(B|\underline{\theta}_T, \underline{\theta}_B)] \\
& + (1 - p_B) (\Delta\theta_T - \Delta\theta_B) [\phi(B, \bar{z}|\bar{\theta}_T, \underline{\theta}_B) - \mathbb{I}(\phi)\phi(B|\underline{\theta}_T, \underline{\theta}_B)] \geq 0.
\end{aligned} \tag{\widetilde{IC}(\underline{\theta}_T)}$$

Note that the controls  $\phi(\cdot|\boldsymbol{\theta})$  associated with the states  $\boldsymbol{\theta} = (\bar{\theta}_T, \bar{\theta}_B)$  and  $\boldsymbol{\theta} = (\bar{\theta}_T, \underline{\theta}_B)$  are linked to the controls associated with the other two states  $\boldsymbol{\theta} = (\underline{\theta}_T, \bar{\theta}_B)$ ,  $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$  only through the constraints  $\widetilde{IC}(\underline{\theta}_T)$  and  $\widetilde{IC}(\underline{\theta}_B)$ . In what follows, we disregard  $\widetilde{IC}(\underline{\theta}_T)$  since it never binds at the optimum. Also note that it is always optimal to set  $\phi^*(T|\boldsymbol{\theta}) = 1$  for  $\boldsymbol{\theta} = (\bar{\theta}_T, \bar{\theta}_B)$  and  $\boldsymbol{\theta} = (\bar{\theta}_T, \underline{\theta}_B)$ . Indeed, this maximizes  $U_S$  and it helps relaxing  $\widetilde{IC}(\underline{\theta}_B)$ .<sup>8</sup>

To derive the optimal mechanism, it thus suffices to consider the monopolist's payoff in the other two states  $\boldsymbol{\theta} = (\underline{\theta}_T, \bar{\theta}_B)$  and  $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$ .

- Consider first  $J \geq 1$ .

1. Suppose  $\phi^* \in \Phi_2$ . Then constraint  $C_2$  necessarily binds since otherwise  $S$  could reduce  $\phi(B|\underline{\theta}_T, \underline{\theta}_B)$  and increase  $\phi(T|\underline{\theta}_T, \underline{\theta}_B)$  enhancing her payoff. Furthermore, when  $\phi(B|\underline{\theta}_T, \underline{\theta}_B) = J\phi(B|\underline{\theta}_T, \bar{\theta}_B)$ ,

$$(1 - p_T) \left\{ -p_B \left[ \frac{p_T}{1-p_T} (\bar{\theta}_T - \bar{\theta}_B) \right] + (1 - p_B) \left[ \frac{p_T}{1-p_T} \Delta\theta_B \right] J \right\} \phi(B|\underline{\theta}_T, \bar{\theta}_B) = 0$$

which implies that  $S$  is indifferent between inducing  $\bar{\theta}_T$  to offer a high or a low price off-equilibrium. Without loss of generality, the optimal mechanism thus belongs to  $\Phi_1$ .

2. When  $\bar{\theta}_B - \frac{p_T}{1-p_T} (\bar{\theta}_T - \bar{\theta}_B) \geq 0$ ,  $\phi^*(B|\underline{\theta}_T, \bar{\theta}_B) = 1$  is clearly optimal. In this case constraints  $C_1$  and  $\widetilde{IC}(\underline{\theta}_B)$  are always satisfied. As for  $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$ , if

$$M(\underline{\theta}_T) + \left( \frac{p_T}{1-p_T} - \frac{p_B}{1-p_B} \right) \Delta\theta_B \geq \max\{0; M(\underline{\theta}_T)\},$$

then  $\phi^*(B|\underline{\theta}_T, \underline{\theta}_B) = 1$  in which case the revenue is  $U_S = (1 - p_B)\underline{\theta}_T + p_T\Delta\theta_B + p_B\underline{\theta}_B$ . If instead

$$M(\underline{\theta}_T) > \max\left\{0; M(\underline{\theta}_T) + \left( \frac{p_T}{1-p_T} - \frac{p_B}{1-p_B} \right) \Delta\theta_B\right\},$$

then  $\phi^*(T|\underline{\theta}_T, \underline{\theta}_B) = 1$  and the revenue is  $(1 - p_B)\underline{\theta}_T + p_B\bar{\theta}_B$ . Finally, if

$$\max\left\{M(\underline{\theta}_T) + \left( \frac{p_T}{1-p_T} - \frac{p_B}{1-p_B} \right) \Delta\theta_B; M(\underline{\theta}_T)\right\} < 0,$$

---

<sup>8</sup>Note that  $\phi(T|\boldsymbol{\theta}) = 1$  is payoff equivalent to  $\phi(B, \bar{z}|\boldsymbol{\theta}) = 1$  for  $\boldsymbol{\theta} = (\bar{\theta}_T, \bar{\theta}_B)$ , and  $\boldsymbol{\theta} = (\bar{\theta}_T, \underline{\theta}_B)$ . Nevertheless, selling to  $T$  in these two states is more effective in relaxing  $\widetilde{IC}(\underline{\theta}_T)$  than selling to  $B$ . This also implies that when  $\widetilde{IC}(\underline{\theta}_T)$  does not bind, the optimal allocation rule need not be unique.

then  $S$  retains the good when  $\theta = (\underline{\theta}_T, \underline{\theta}_B)$  and the revenue is  $U_S = p_T(1 - p_B)\bar{\theta}_T + p_B\bar{\theta}_B$ .<sup>9</sup>

Next, assume  $\bar{\theta}_B - \frac{p_T}{1-p_T}(\bar{\theta}_T - \bar{\theta}_B) < 0$ . In this case  $\widetilde{IC}(\underline{\theta}_B)$  necessarily binds, i.e.

$$\phi^*(B|\underline{\theta}_T, \bar{\theta}_B) = \phi^*(B|\underline{\theta}_T, \underline{\theta}_B),$$

and hence  $C_1$  is always satisfied. Furthermore, since  $M(\underline{\theta}_T) \leq \bar{\theta}_B - \frac{p_T}{1-p_T}(\bar{\theta}_T - \bar{\theta}_B) < 0$ ,  $S$  never sells to  $T$  when the latter reports a low valuation, i.e. when  $\theta = (\underline{\theta}_T, \bar{\theta}_B)$  or  $\theta = (\underline{\theta}_T, \underline{\theta}_B)$ . At the optimum  $\phi^*(B|\underline{\theta}_T, \bar{\theta}_B) = \phi^*(B|\underline{\theta}_T, \underline{\theta}_B) = 1$  if

$$p_B \left[ \bar{\theta}_B - \frac{p_T}{1-p_T}(\bar{\theta}_T - \bar{\theta}_B) \right] + (1 - p_B) \left[ M(\underline{\theta}_T) + \left( \frac{p_T}{1-p_T} - \frac{p_B}{1-p_B} \right) \Delta\theta_B \right] \geq 0$$

and  $\phi^*(B|\underline{\theta}_T, \underline{\theta}_B) = \phi^*(B|\underline{\theta}_T, \underline{\theta}_B) = 0$  otherwise. In the first case, the revenue is  $U_S = (1 - p_B)\underline{\theta}_T + p_T\Delta\theta_B + p_B\underline{\theta}_B$ , whereas in the second  $U_S = p_T\bar{\theta}_T$ .

• Suppose now  $J < 1$ .

1. In this case  $\widetilde{IC}(\underline{\theta}_B)$  can be neglected as it is never binding at the optimum. Furthermore, without loss of generality, the optimal mechanism belongs to  $\Phi_1$ . The argument is the same as for  $J \geq 1$ .
2. Assume now  $\bar{\theta}_B - \frac{p_T}{1-p_T}(\bar{\theta}_T - \bar{\theta}_B) \geq 0$ . Then  $\phi^*(B|\underline{\theta}_T, \bar{\theta}_B) = 1$ . As for  $\theta = (\underline{\theta}_T, \underline{\theta}_B)$ , if

$$M(\underline{\theta}_T) + \left( \frac{p_T}{1-p_T} - \frac{p_B}{1-p_B} \right) \Delta\theta_B \geq \max\{0, M(\underline{\theta}_T)\}$$

then constraint  $C_1$  binds and thus  $\phi^*(B|\underline{\theta}_T, \underline{\theta}_B) = J$ . If in addition  $M(\underline{\theta}_T) \geq 0$ , then  $\phi^*(T|\underline{\theta}_T, \underline{\theta}_B) = 1 - J$ ; otherwise,  $\phi^*(T|\underline{\theta}_T, \underline{\theta}_B) = 0$ . In the former case, the expected revenue is  $\frac{p_B}{1-p_B}[\bar{\theta}_T(p_T - p_B) + (1 - p_T)\bar{\theta}_B] + \underline{\theta}_T(1 - p_B)$ , whereas in the latter  $(1 - p_B)\underline{\theta}_T + p_T\Delta\theta_B + p_B\underline{\theta}_B$ . If, on the contrary,

$$M(\underline{\theta}_T) + \left( \frac{p_T}{1-p_T} - \frac{p_B}{1-p_B} \right) \Delta\theta_B < \max\{0, M(\underline{\theta}_T)\},$$

then necessarily  $\phi^*(B|\underline{\theta}_T, \underline{\theta}_B) = 0$ . As for  $\phi(T|\underline{\theta}_T, \underline{\theta}_B)$ , at the optimum  $\phi^*(T|\underline{\theta}_T, \underline{\theta}_B) = 1$  when  $M(\underline{\theta}_T) \geq 0$ , whereas  $\phi^*(T|\underline{\theta}_T, \underline{\theta}_B) = 0$  when  $M(\underline{\theta}_T) < 0$ . The revenue is equal to  $(1 - p_B)\underline{\theta}_T + p_B\bar{\theta}_B$  in the first case and  $(1 - p_B)\underline{\theta}_T + p_T\Delta\theta_B + p_B\underline{\theta}_B$  in the second. Next, consider  $\bar{\theta}_B - \frac{p_T}{1-p_T}(\bar{\theta}_T - \bar{\theta}_B) < 0$ . At the optimum, constraint  $C_1$  necessarily binds. It follows that  $\phi^*(B|\underline{\theta}_T, \bar{\theta}_B) = 1$  and  $\phi^*(B|\underline{\theta}_T, \underline{\theta}_B) = J$  if

$$p_B \left[ \bar{\theta}_B - \frac{p_T}{1-p_T}(\bar{\theta}_T - \bar{\theta}_B) \right] + (1 - p_B)J \left[ M(\underline{\theta}_T) + \left( \frac{p_T}{1-p_T} - \frac{p_B}{1-p_B} \right) \Delta\theta_B \right] > 0, \quad (2)$$

---

<sup>9</sup>Again, the solution may not be unique, as  $\phi(T|\theta) = 1$  is payoff equivalent to  $\phi(B|\theta) = 1$  for  $\theta = (\underline{\theta}_T, \bar{\theta}_B)$ . For example, if  $M(\underline{\theta}_T) > \max\left\{0; M(\underline{\theta}_T) + \left( \frac{p_T}{1-p_T} - \frac{p_B}{1-p_B} \right) \Delta\theta_B\right\}$ , then  $\phi^*(T|\underline{\theta}_T, \bar{\theta}_B) = 1$  is also optimal.

whereas  $\phi^*(B|\underline{\theta}_T, \bar{\theta}_B) = \phi^*(B|\underline{\theta}_T, \underline{\theta}_B) = 0$  when (2) is reversed. In either case,  $S$  never sells to  $T$  when the latter reports a low valuation, that is,  $\phi^*(T|\theta) = 0$  when  $\theta = (\underline{\theta}_T, \bar{\theta}_B)$  and  $\theta = (\underline{\theta}_T, \underline{\theta}_B)$ . The revenue is

$$p_T \bar{\theta}_T + p_B(\bar{\theta}_B - p_T \bar{\theta}_T) + (1 - p_T)(1 - p_B) \left[ M(\underline{\theta}_T) + \left( \frac{p_T}{1 - p_T} - \frac{p_B}{1 - p_B} \right) \Delta \theta_B \right] J$$

in the former case, and  $p_T \bar{\theta}_T$  in the latter.

**$B$  has all bargaining power (i.e.  $\lambda_B = 1$ ).**

In this case,  $S$  does not need to disclose any information to  $T$ . Therefore, we eliminate  $z_T$  from the mechanism  $\phi$ . Furthermore, since  $\underline{\theta}_B \leq \underline{\theta}_T \leq \bar{\theta}_B \leq \bar{\theta}_T$ , when  $h = T$ , the only incentive-compatible recommendation is for  $\underline{\theta}_B$  to offer  $t^r(\underline{\theta}_B) \leq \underline{\theta}_B$  and for  $\bar{\theta}_B$  to offer  $t^r(\bar{\theta}_B) = \underline{\theta}_T$  if  $\Pr(\underline{\theta}_T|\cdot) > 0$  and any  $t^r \leq \bar{\theta}_B$  otherwise. Similarly, when  $h = B$ , the only incentive-compatible recommendation for  $\bar{\theta}_B$  is to ask a price  $t^r(\bar{\theta}_B) = \bar{\theta}_T$  if  $\Pr(\bar{\theta}_T|\cdot) > 0$  and any price  $t^r \geq \bar{\theta}_B$  otherwise. Without loss of generality, we will assume that  $\bar{\theta}_B$  always asks  $t^r(\bar{\theta}_B) = \bar{\theta}_T$  when  $h = B$  and offers  $t^r(\bar{\theta}_B) = \underline{\theta}_T$  when  $h = T$ . Since these are the same prices that  $B$  offers in the absence of any explicit recommendation, to save on notation, we will drop  $z_B$  from the mapping  $\phi$  when  $h = T$ , or  $h = B$  and  $\theta_B = \bar{\theta}_B$ . When instead  $h = B$  and  $\theta_B = \underline{\theta}_B$ , we will denote by  $\bar{z}$  and  $\underline{z}$  the recommendations (for  $\underline{\theta}_B$ ) to ask  $t^r(\underline{\theta}_B) = \bar{\theta}_T$  and  $t^r(\underline{\theta}_B) = \underline{\theta}_T$ , respectively. For these recommendations to be incentive compatible, the mechanism  $\phi$  must satisfy

$$\begin{aligned} \phi(B, \underline{z}|\underline{\theta}_T, \underline{\theta}_B) &\geq Q\phi(B, \underline{z}|\bar{\theta}_T, \underline{\theta}_B) & \widetilde{IC}(\underline{z}, \underline{\theta}_B) \\ \phi(B, \bar{z}|\underline{\theta}_T, \underline{\theta}_B) &\leq Q\phi(B, \bar{z}|\bar{\theta}_T, \underline{\theta}_B) & \widetilde{IC}(\bar{z}, \underline{\theta}_B) \end{aligned}$$

where  $Q := \frac{p_T \Delta \theta_T}{(1 - p_T)(\underline{\theta}_T - \underline{\theta}_B)}$ . The maximal revenue for the monopolist can be derived by partitioning the set of direct mechanisms  $\Phi$  into two classes. The first one, which we denote by  $\Phi_1$ , is such that  $\underline{\theta}_B$  asks a price  $t^r(\underline{\theta}_B) = \bar{\theta}_T$  off-equilibrium, after reporting  $\theta_B = \bar{\theta}_B$ . The second,  $\Phi_2$ , is such that  $\underline{\theta}_B$  asks a price  $t^r(\underline{\theta}_B) = \underline{\theta}_T$ . For a mechanism  $\phi$  to belong to  $\Phi_1$  it must be that

$$\phi(B|\underline{\theta}_T, \bar{\theta}_B) \leq Q\phi(B|\bar{\theta}_T, \bar{\theta}_B) \quad C_1$$

Similarly, if  $\phi \in \Phi_2$ , then

$$\phi(B|\underline{\theta}_T, \bar{\theta}_B) \geq Q\phi(B|\bar{\theta}_T, \bar{\theta}_B) \quad C_2$$

Let  $\mathbb{I}(\phi) = 1$  if  $\phi \in \Phi_1$  and  $\mathbb{I}(\phi) = 0$  if  $\phi \in \Phi_2$ . Substituting the values of  $s_T(\cdot)$  and  $r_B(\cdot)$ , the

program for the optimal mechanism reduces to

$$\begin{aligned}
U_S = & p_T p_B \left\{ \bar{\theta}_T [\phi(T|\bar{\theta}_T, \bar{\theta}_B) + \phi(B|\bar{\theta}_T, \bar{\theta}_B)] \right\} + \\
& + p_T (1 - p_B) \left\{ \bar{\theta}_T [\phi(T|\bar{\theta}_T, \underline{\theta}_B) + \phi(B, \bar{z}|\bar{\theta}_T, \underline{\theta}_B)] \right\} + \\
& + \left[ \bar{\theta}_T - \frac{p_B}{1-p_B} \Delta \theta_T \right] \phi(B, \bar{z}|\bar{\theta}_T, \underline{\theta}_B) \left\} + \\
& + (1 - p_T) p_B \left\{ \left[ \bar{\theta}_B - \frac{p_T}{1-p_T} \Delta \theta_T \right] \phi(T|\underline{\theta}_T, \bar{\theta}_B) + \bar{\theta}_B \phi(B|\underline{\theta}_T, \bar{\theta}_B) \right\} + \\
& + (1 - p_T) (1 - p_B) \left\{ M(\underline{\theta}_B) \phi(B, \bar{z}|\underline{\theta}_T, \underline{\theta}_B) + \right. \\
& \left. + \left[ M(\underline{\theta}_T) - \frac{p_B}{1-p_B} (\bar{\theta}_B - \underline{\theta}_T) \right] [\phi(T|\underline{\theta}_T, \underline{\theta}_B) + \phi(B, \bar{z}|\underline{\theta}_T, \underline{\theta}_B)] \right\},
\end{aligned}$$

subject to  $\widetilde{IC}(\bar{z}, \underline{\theta}_B)$ ,  $\widetilde{IC}(\bar{z}, \underline{\theta}_B)$ ,  $C_1$ ,  $C_2$ , and

$$\begin{aligned}
p_B [\phi(T|\bar{\theta}_T, \bar{\theta}_B) - \phi(T|\underline{\theta}_T, \bar{\theta}_B)] + (1 - p_B) [\phi(T|\bar{\theta}_T, \underline{\theta}_B) - \phi(T|\underline{\theta}_T, \underline{\theta}_B)] \\
+ (1 - p_B) [\phi(B, \bar{z}|\bar{\theta}_T, \underline{\theta}_B) - \phi(B, \bar{z}|\underline{\theta}_T, \underline{\theta}_B)] \geq 0, \tag{\widetilde{IC}(\underline{\theta}_T)}
\end{aligned}$$

$$\begin{aligned}
p_T \Delta \theta_T [1 - \mathbb{I}(\phi)] [\phi(B|\bar{\theta}_T, \bar{\theta}_B) - \phi(B|\bar{\theta}_T, \underline{\theta}_B)] + \\
+ (1 - p_T) \Delta \theta_B \mathbb{I}(\phi) [\phi(B|\underline{\theta}_T, \bar{\theta}_B) - \phi(B|\underline{\theta}_T, \underline{\theta}_B)] + \\
+ (1 - p_T) (\bar{\theta}_B - \underline{\theta}_T) [\phi(T|\underline{\theta}_T, \bar{\theta}_B) - \phi(T|\underline{\theta}_T, \underline{\theta}_B)] + \\
+ (1 - p_T) (\bar{\theta}_B - \underline{\theta}_T) [[1 - \mathbb{I}(\phi)] \phi(B|\underline{\theta}_T, \bar{\theta}_B) - \phi(B, \bar{z}|\underline{\theta}_T, \underline{\theta}_B)] \geq 0, \tag{\widetilde{IC}(\underline{\theta}_B)}
\end{aligned}$$

Note that  $U_S$  is the same if the mechanism belongs to  $\Phi_1$  or  $\Phi_2$ . Thus, the discrimination over the two cases occurs uniquely on the basis of the constraints.

- Assume first  $\max \left\{ M(\underline{\theta}_T) - \frac{p_B}{1-p_B} (\bar{\theta}_B - \underline{\theta}_T); M(\underline{\theta}_B) \right\} \geq 0$ .
  1. If  $M(\underline{\theta}_T) - \frac{p_B}{1-p_B} (\bar{\theta}_B - \underline{\theta}_T) \leq M(\underline{\theta}_B)$ , then  $Q \geq 1$ . In this case the mechanism  $\phi^*(B|\theta) = 1$  for  $\theta = (\underline{\theta}_T, \bar{\theta}_B)$ ,  $\theta = (\bar{\theta}_T, \bar{\theta}_B)$  and  $\phi^*(B, \bar{z}|\theta) = 1$  for  $\theta = (\underline{\theta}_T, \underline{\theta}_B)$ ,  $\theta = (\bar{\theta}_T, \underline{\theta}_B)$  which belongs to  $\Phi_1$  maximizes  $U_S$ , and satisfies all constraints. For any  $\theta_B$ ,  $B$  always asks  $t^r(\theta_B) = \bar{\theta}_T$  and thus trade occurs in the secondary market if and only if  $T$  has a high valuation. In this case, the final allocation and the expected revenue coincide with that in the Myerson optimal auction if  $M(\underline{\theta}_T) \leq M(\underline{\theta}_B)$ . If, on the contrary,  $M(\underline{\theta}_T) > M(\underline{\theta}_B)$ , then in state  $\theta = (\underline{\theta}_T, \underline{\theta}_B)$ ,  $B$  retains the good, contrary to what prescribed by the Myerson allocation rule. This in turn induces a loss of expected revenue equal to  $(1 - p_T)(1 - p_B) [M(\underline{\theta}_T) - M(\underline{\theta}_B)]$ .
  2. If  $M(\underline{\theta}_T) - \frac{p_B}{1-p_B} (\bar{\theta}_B - \underline{\theta}_T) > M(\underline{\theta}_B)$ , the following mechanism which belongs to  $\Phi_2$  maximizes  $U_S$  and satisfies all constraints

$$\phi^*(T|\bar{\theta}_T, \underline{\theta}_B) = \phi^*(T|\underline{\theta}_T, \underline{\theta}_B) = \phi^*(T|\bar{\theta}_T, \bar{\theta}_B) = \phi^*(B|\underline{\theta}_T, \bar{\theta}_B) = 1.$$

Trade does not occur in the secondary market, the final allocation is exactly as in Myerson, but the expected revenue is just  $p_T p_B \bar{\theta}_T + (1 - p_T p_B) \underline{\theta}_T$  instead of

$$\mathbb{E}_\theta [\max \{0, M(\theta_T), M(\theta_B)\}] = p_B [p_T \bar{\theta}_T + (1 - p_T) \bar{\theta}_B] + (1 - p_B) \underline{\theta}_T.$$

- Assume now  $\max \left\{ M(\underline{\theta}_T) - \frac{p_B}{1-p_B}(\bar{\theta}_B - \underline{\theta}_T); M(\underline{\theta}_B) \right\} < 0$ . In this case,  $S$  finds it optimal to retain the good when  $\theta = (\underline{\theta}_T, \underline{\theta}_B)$ . As for the other states, the following mechanism, which belongs to  $\Phi_2$ , maximizes  $U_S$  and satisfies all constraints

$$\phi^*(T|\bar{\theta}_T, \bar{\theta}_B) = \phi^*(T|\bar{\theta}_T, \underline{\theta}_B) = \phi^*(B|\underline{\theta}_T, \bar{\theta}_B) = 1.$$

The monopolist's expected revenue is  $p_T \bar{\theta}_T + (1 - p_T) p_B \bar{\theta}_B$ , trade occurs in the primary market if and only if at least one of the two bidders has a high valuation, and no offers are made in the resale game. If  $M(\underline{\theta}_T) \leq 0$ , the expected revenue is the same as in the Myerson auction. On the contrary, if  $M(\underline{\theta}_T) > 0 > M(\underline{\theta}_B)$ ,  $S$  incurs a loss equal to  $(1 - p_T)(1 - p_B)M(\underline{\theta}_T)$ .

We conclude that when  $\lambda_B = 1$ , the impossibility to prohibit resale results in a loss of expected revenue for the monopolist if and only if  $M(\underline{\theta}_T) > \max\{0, M(\underline{\theta}_B)\}$ . ■

## References

- [1] Maskin, E., and J. Tirole, (1990): "The Principal-Agent Relationship with an Informed Principal: The Case of Private Values," *Econometrica*, Vo. 58, 379–409.