Web appendix of the paper "Competitive quantity discounts" by Giacomo Calzolari and Vincenzo Denicolò, Nov. 4th 2011

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(*The derivation of the analysis has been grouped in
 the following cells that could be opened to see the content*)


(*PART I EQUILIBRIUM DERIVATION  WITH NON-LINEAR  PRICES*)

(*HERE WE FIRST DETERMINE EQUILIBRIUM NON-LINEAR PRICES*)

(*PREFERENCES*)
u = x (qa + qb) - (1 - g) / 2 (qa^2 + qb^2) - g qa qb;
(*types x uniformly distributed on [0,1] *)
ca = 0;
cb = c * qb;

uesc = u /. qb → 0;

(*General linear-quadratic schedules*)
pa = a0 + a1 qa + a2 qa^2;
pb = b1 qb + b2 qb^2;

(*notice that to clarify notation here we indicate the parameters
 for A's tariff with a0, a1, a2  and those of B with b0, b1, b2
 Furthermore, b0=0 for Wilson's Lemma*)
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```
(*UNCONSTRAINED MONOPOLY*)
qm = qa /. Solve[D[u - pa /. qb → 0 , qa] == 0, qa][[1, 1]]

(*for no distortion at the top it must be the following ,
that allows to determine the a2*)
tmp2 = qa /. Solve[D[u /. qb → 0 , qa] == 0, qa][[1, 1]]
a2m = a2 /. FullSimplify[Solve[qm == tmp2 /. x → 1, a2]][[1, 1]]

(*then define profits*)
Prm = FullSimplify[Integrate[pa /. qa → qm /. a2 → a2m, {x, a1, 1}]]
```

$$\frac{-a1 + x}{1 + 2\,a2 - g}$$

$$-\frac{x}{-1 + g}$$

$$\frac{1}{2}\,a1\,(-1 + g)$$

$$\frac{(-1 + a1)\,(a1 - 3\,a0\,(-1 + g))}{3\,(-1 + g)}$$

```
(*and find the optimal a1*)
(*notice that the uncontrained mopolist makes a certain type indifferent to
  participating and not and then for Wilson Lemma it must be that a0=0, hence ...*)

tmp = a1 /. FullSimplify[Solve[{D[Prm, a1] == 0 /. a0 → 0}, {a1}]][[1, 1]]

a1m = FullSimplify[tmp]
a2m = a2m /. a1 → a1m
```

$$\frac{1}{2}$$

$$\frac{1}{2}$$

$$\frac{1}{4}\,(-1 + g)$$

```
(*hence we can define the pricing schedule with uncontrainted mopoly*)
pm = a0 + a1m qa + a2m qa^2 /. a0 → 0
qm = FullSimplify[qm /. {a1 → a1m, a2 → a2m}]
```

$$\frac{qa}{2} + \frac{1}{4}\,(-1 + g)\,qa^2$$

$$\frac{1 - 2\,x}{-1 + g}$$

```
(*CONSTRAINED MONOPOLIST*)
```

```
(*Limit Price when B is not active*)

qmlim = qa /. FullSimplify[Solve[c == D[u, qb] /. qb → 0, qa]][[1, 1]]

(*which is implemented identifying the parameters a1 and a2 such that qm=qmlim*)
tmp = FullSimplify[Solve[{-a1/(1 + 2 a2 - g) == -c/g, 1/(1 + 2 a2 - g) == 1/g}, {a1, a2}]]
a1mlim = a1 /. tmp[[1, 1]]
a2mlim = a2 /. tmp[[1, 2]]
```

$$\frac{-c + x}{g}$$

$$\left\{\left\{a1 \to c,\ a2 \to -\frac{1}{2} + g\right\}\right\}$$

$$c$$

$$-\frac{1}{2} + g$$

```
(*Here we verify when the mopolist is indeed unconstrained,
i.e. when it is the case that for any x qm>qalim:*)
cd = c /. FullSimplify[Solve[(qm /. x → 1 ) == qmlim /. x → 1, c]][[1, 1]]
Factor[qm - qmlim /. x → 1]
```

$$2 + \frac{1}{-1 + g}$$

$$\frac{1 - c - 2 g + c g}{(-1 + g)\ g}$$

```
(*notice that cd could be found also as in
  the paper: the c such that the marginal utility for type x=
 1 of good qb when qa is the monopoly quantity is equal to c,
so that B can never be active fi it has a larger c*)

FullSimplify[Solve[0 == D[u, qb] - c /. {qb → 0, qa → qm} /. x → 1, c]]
```

$$\left\{\left\{c \to 2 + \frac{1}{-1 + g}\right\}\right\}$$

```
(*hence, for c≥cd qm>qmlim for the highest typex =1*)

(*We now see that for g<1/3 qmlim is steeper in x than qm and viceversa for g≥1/3*)
FullSimplify[D[qm, x] - D[qmlim, x]]
```

$$\frac{1 - 3 g}{(-1 + g)\ g}$$

```
(*Hence it follows that for g<1/3 when c≥
  cd qm≥qmlim for any x and the mopolist is uncontested
   we then explore the case g≥1/3  now...*)
```

```
xm = x /. Solve[qmlim == qm, x][[1, 1]]
FullSimplify[Solve[1 / 2 == xm , c]]
FullSimplify[xm - 1 / 2]
```

$$\frac{-c + g + c\,g}{-1 + 3\,g}$$

$$\left\{\left\{c \to \frac{1}{2}\right\}\right\}$$

$$\frac{(-1 + 2\,c)\,(-1 + g)}{-2 + 6\,g}$$

```
(*hence when g≥1/3 (i) if c>1/2 then qmlim and qm intersect at a xm<
  1/2 where 1/2 is the lowest buying type with qm and then since
   qm is steeper than qalim (see above) we again have that  qm≥
  qmlim for any x and the mopolist is uncontested
   (ii) if instead c<1/2 for x<xm qm<qmlim and then the quantity that can be
   implemented for those types is qmlim and the lowest participating type is x=
 c whilst the loest type buying with qm is xm. The proof in the
  text shows that indeed qmlim is optimal in this case*)


(*Summarizing the MONOPOLY CASES are as follows:*)

(*Region RM: uncontested mopoly*)
(*the other parameters a1 and a2 determined as above
 as well as qm and then here we have by Wilson's Lemma*)
a0RM = 0;
a1RM = a1m;
a2RM = a2m;
paRM = pm;
qaRM = qm;

pm /. qa → qm;
PraRM = FullSimplify[Integrate[%, {x, 1 / 2, 1}]]
(*the marginal type is at x=1/2, see qamon*)
PrbRM = 0;

uRM = FullSimplify[uesc - pm /. qa → qm]
URM = FullSimplify[Integrate[uRM, {x, 1 / 2, 1}]]

WRM = FullSimplify[PraRM + PrbRM + URM]

CarltonRM = Max[PrbRM, URM]
```

$$\frac{1}{12 - 12\,g}$$

$$-\frac{(1 - 2\,x)^2}{4\,(-1 + g)}$$

$$\frac{1}{24 - 24\,g}$$

$$\frac{1}{8 - 8\,g}$$

$$\mathrm{Max}\left[0,\ \frac{1}{24 - 24\,g}\right]$$

```
(*Region
 RML: for high types there is monopoly and for low types there is limit pricing,
let us derive all parameters here:
  the idea is that qm, a1, a2 remain unchanged
 (notice that this implies that the a1 is still determined under the assumption a0=0),
the only difference is that a0 is no more nil, whilst a0mlim=0 for Wilson's lemma;
a0m is determined by making the prices pm and pamlim equal for the indifferent type xm
*)


paRLMlow = pa /. {a1 → a1mlim, a2 → a2mlim, a0 → 0};

tmp =
 FullSimplify[Solve[(paRLMlow /. qa -> qmlim /. x -> xm) == (a0 + pm /. qa -> qm /. x -> xm), a0]]
a0RLMhigh = a0 /. tmp[[1, 1]]
(*and for types larger than xm we have as in the uncontrained mopolist*)
a1RLMhigh = a1m;
a2RLMhigh = a2m;
qaRLMhigh = qm;
paRLMhigh = pa /. {a0 -> a0RLMhigh, a1 -> a1RLMhigh, a2 -> a2RLMhigh};

xRLM = xm;

(*for lower types we have the constrained quantity and schedule:*)
a0RLMlow = 0;
a1RLMlow = a1mlim;
a2RLMlow = a2mlim;
qaRLMlow = qmlim;
(*the schedule paRLMlow is defined few lines above*)
```

$$\left\{\left\{a0 \to -\frac{(1-2c)^2}{-4+12g}\right\}\right\}$$

$$-\frac{(1-2c)^2}{-4+12g}$$

```
(*Defining the payoffs*)
pa /. {a1 → a1RLMlow, a2 -> a2RLMlow, a0 -> a0RLMlow} /. qa → qmlim;
pm /. qa -> qm;

PraRLM = FullSimplify[Integrate[%%, {x, c, xRLM}] + Integrate[% + a0RLMhigh, {x, xRLM, 1}]]
PrbRLM = 0

uesc - pa /. {a1 → a1RLMlow, a2 -> a2RLMlow, a0 -> a0RLMlow} /. qa → qmlim;
uesc - pm - a0RLMhigh /. qa -> qm;
URLM = FullSimplify[Integrate[%%, {x, c, xRLM}] + Integrate[%, {x, xRLM, 1}]]

WRLM = FullSimplify[URLM + PraRLM + PrbRLM ]

CarltonRLM = Max[PrbRLM, URLM];
```

$$\frac{1 + c\left(3 - 6c + 4c^2\right)(-1 + g) - 2g}{6(1 + g(-4 + 3g))}$$

0

$$\frac{1 + (-3 + g)g + 3c^2(-1 + g)(-3 + 7g) - 4c^3(1 + g(-3 + 2g)) - 3c(2 + g(-7 + 5g))}{6(1 - 3g)^2(-1 + g)}$$

$$\frac{3(-1 + c)c + 2g + c(9 - 2c(3 + 2c))g + (1 + c)^2(-5 + 4c)g^2}{6(1 - 3g)^2(-1 + g)}$$

```
(*DUOPOLY*)

(*DUOPOLY: DERIVATION OF PRICE SCHEDULES*)

(*Consumer's quantities in duopoly*)
tmp = FullSimplify[Solve[{D[u - pa - pb, qa] == 0, D[u - pa - pb, qb] == 0}, {qa, qb}]]
qad = qa /. tmp[[1, 1]]
qbd = qb /. tmp[[1, 2]]
```

$$\left\{\left\{qa \to \frac{b1\,g + a1(-1 - 2b2 + g) + x + 2b2\,x - 2g\,x}{(1 + 2a2)(1 + 2b2) - 2(1 + a2 + b2)g},\ qb \to \frac{a1\,g + b1(-1 - 2a2 + g) + x + 2a2\,x - 2g\,x}{(1 + 2a2)(1 + 2b2) - 2(1 + a2 + b2)g}\right\}\right\}$$

$$\frac{b1\,g + a1(-1 - 2b2 + g) + x + 2b2\,x - 2g\,x}{(1 + 2a2)(1 + 2b2) - 2(1 + a2 + b2)g}$$

$$\frac{a1\,g + b1(-1 - 2a2 + g) + x + 2a2\,x - 2g\,x}{(1 + 2a2)(1 + 2b2) - 2(1 + a2 + b2)g}$$

```
(*using no-distortion at the top to determine a2 and b2*)
tmp = FullSimplify[Solve[{D[u, qa] == 0, D[u - c qb, qb] == 0}, {qa, qb}]]
qafb = qa /. tmp[[1, 1]]
qbfb = qb /. tmp[[1, 2]]

tmp = FullSimplify[Solve[{qafb == qad /. x → 1, qbfb == qbd /. x → 1}, {a2, b2}]]
a2d = a2 /. tmp[[1, 1]]
b2d = b2 /. tmp[[1, 2]]
```

$$\left\{\left\{qa \to \frac{c\,g}{1-2\,g}+x,\ qb \to \frac{c-c\,g}{-1+2\,g}+x\right\}\right\}$$

$$\frac{c\,g}{1-2\,g}+x$$

$$\frac{c-c\,g}{-1+2\,g}+x$$

$$\left\{\left\{a2 \to \frac{a1\,(-1+2\,g)}{2+2\,(-2+c)\,g},\ b2 \to \frac{(b1-c)\,(-1+2\,g)}{2+2\,c\,(-1+g)-4\,g}\right\}\right\}$$

$$\frac{a1\,(-1+2\,g)}{2+2\,(-2+c)\,g}$$

$$\frac{(b1-c)\,(-1+2\,g)}{2+2\,c\,(-1+g)-4\,g}$$

```
(*To derive the duopoly schedules,
we have to procede keeping into account that a1 and b1 each affect
 participation under the duopoly scheduels. To this end we procede as follows:*)

(*first define for a given type t,
its associated indirect utility when buying from both firms*)
vt = FullSimplify[u - pa - pb - b0 /. {qa → qad, qb → qbd} /. x -> t]
```

$$\frac{1}{2\,(1+2\,a2)\,(1+2\,b2)-4\,(1+a2+b2)\,g}\Big(-2\,a0\,(1+2\,a2)\,(1+2\,b2)-2\,(1+2\,a2)\,b0\,(1+2\,b2)+b1^2\,(1+2\,a2-g)+ a1^2\,(1+2\,b2-g)+4\,a0\,(1+a2+b2)\,g+4\,b0\,(1+a2+b2)\,g- 2\,b1\,(1+2\,a2-2\,g)\,t+2\,(1+a2+b2-2\,g)\,t^2-2\,a1\,(b1\,g+t+2\,b2\,t-2\,g\,t)\Big)$$

```
(*Then we solve for the fixed fees (in equilibrium could be nil)
 leaveing a degree of freedom over the level of utility type t gets: i.e. *)

a0dtmp = a0 /. Solve[vt == k, a0][[1, 1]];
b0dtmp = b0 /. Solve[vt == h, b0][[1, 1]];

(*Then we consider the uppart part of profits, that with duopoly, *)

profitatmp = FullSimplify[a1 qa + a2 qa^2 + a0dtmp /. {qa → qad} /. {a2 → a2d, b2 → b2d}];
profitbtmp = FullSimplify[b1 qb + b2 qb^2 - c qb + b0dtmp /. {qb → qbd} /. {a2 → a2d, b2 → b2d}];
```

```
profita = Integrate[profitatmp, {x, t, 1}];
profitb = Integrate[profitbtmp, {x, t, 1}];

Ha1 = D[profita, a1];
Hb1 = D[profitb, b1];
Simplify[{Ha1 == 0, Hb1 == 0}];
solutionsd = FullSimplify[Solve[%, {a1, b1}]]
```

$$\left\{\left\{a1 \to 1+(-2+c)\, g,\ b1 \to \frac{-1+2 g (2+(-2+c) g)}{-1+3 g}\right\},\right.$$
$$\left\{a1 \to \frac{(-1+2 g)(1+(-2+c) g)}{-1+3 g},\ b1 \to 1+(-2+c)\, g\right\},$$
$$\left\{a1 \to \frac{(1+(-2+c) g)\left(3-3 g+\sqrt{1+g(-2+9 g)}\right)}{4-8 g},\right.$$
$$\left. b1 \to -\frac{3+c-9 g-2 c g+6 g^2-3 c g^2+(1+c(-1+g)-2 g)\sqrt{1+g(-2+9 g)}}{2(-2+4 g)}\right\},$$
$$\left\{a1 \to \frac{(1+(-2+c) g)\left(-3+3 g+\sqrt{1+g(-2+9 g)}\right)}{-4+8 g},\right.$$
$$\left.\left. b1 \to \frac{-3-c+9 g+2 c g-6 g^2+3 c g^2+(1+c(-1+g)-2 g)\sqrt{1+g(-2+9 g)}}{2(-2+4 g)}\right\}\right\}$$

```
(*Substituting any one of the four pairs of solutions into quantities qad and qbd,
the three first pair of solutions lead to impossibilities and the fourth instead not,
hence the solution is the fourth as it can be also verified noticing that it
  is the only one that coincides with that of the symmetric case when c=0*)

FullSimplify[solutionsd /. c → 0]
```

$$\left\{\left\{a1 \to 1-2 g,\ b1 \to -\frac{(1-2 g)^2}{-1+3 g}\right\},\ \left\{a1 \to -\frac{(1-2 g)^2}{-1+3 g},\ b1 \to 1-2 g\right\},\right.$$
$$\left\{a1 \to \frac{1}{4}\left(3-3 g+\sqrt{1+g(-2+9 g)}\right),\ b1 \to \frac{1}{4}\left(3-3 g+\sqrt{1+g(-2+9 g)}\right)\right\},$$
$$\left.\left\{a1 \to \frac{1}{4}\left(3-3 g-\sqrt{1+g(-2+9 g)}\right),\ b1 \to \frac{1}{4}\left(3-3 g-\sqrt{1+g(-2+9 g)}\right)\right\}\right\}$$

```
(*Before defining a1 and b1 is useful to write
 them in a more compact way based on the symmetric case:*)
```

$$as = \frac{1}{4}\left(3-3 g-\sqrt{1+g(-2+9 g)}\right);$$

```
FullSimplify[a + (c g a / (1 - 2 g)) - a1 /. solutionsd[[4, 1]] /. a → as]
FullSimplify[a + (c (1 - 2 g - a (1 - g)) / (1 - 2 g)) - b1 /. solutionsd[[4, 2]] /. a → as]
```

0

0

```
(*Hence, we (re)define ...*)
a1d = a + (c g a / (1 - 2 g))
b1d = a + (c (1 - 2 g - a (1 - g)) / (1 - 2 g))
a2d = FullSimplify[a2d /. a1 → a1d]
b2d = FullSimplify[b2d /. b1 → b1d]

pad = a0 + a1d qa + a2d qa^2
pbd = b0 + b1d qb + b2d qb^2
```

$$a + \frac{a\,c\,g}{1 - 2\,g}$$

$$a + \frac{c\,(1 - a\,(1 - g) - 2\,g)}{1 - 2\,g}$$

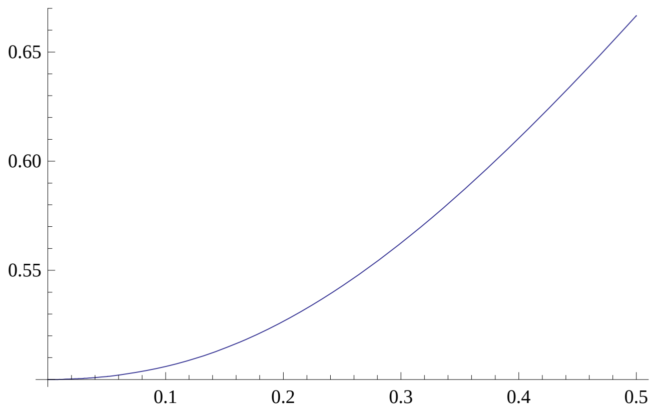
$$-\frac{a}{2}$$

$$-\frac{a}{2}$$

$$a0 + \left(a + \frac{a\,c\,g}{1 - 2\,g}\right) qa - \frac{a\,qa^2}{2}$$

$$b0 + \left(a + \frac{c\,(1 - a\,(1 - g) - 2\,g)}{1 - 2\,g}\right) qb - \frac{a\,qb^2}{2}$$

```
( (1 - 2 g - a (1 - g)) / (1 - 2 g)) /. a → as;
Plot[%, {g, 0, 1 / 2}]
```




```
(*Now we define the LIMIT TARIFF of A when
 B is active and then offers pbd determined above*)
(*Limit Price when B is not active*)
FullSimplify[Solve[b1 == D[u, qb] /. qb → 0, qa]]
qdlim = Factor[qa /. %[[1, 1]] /. b1 → b1d]
```

$$\left\{\left\{qa \to \frac{-b1 + x}{g}\right\}\right\}$$

$$\frac{a + c - a\,c - 2\,a\,g - 2\,c\,g + a\,c\,g - x + 2\,g\,x}{g\,(-1 + 2\,g)}$$

```
(*which is implemented identifying the parameters a1 and a2 such that qm=qmlim*)
tmp = FullSimplify[Solve[{-a1/(1 + 2 a2 - g) == -b1/g, 1/(1 + 2 a2 - g) == 1 / g}, {a1, a2}]]
a1dlim = a1 /. tmp[[1, 1]]
a2dlim = a2 /. tmp[[1, 2]]
```

$$\left\{\left\{a1 \to b1,\ a2 \to -\frac{1}{2} + g\right\}\right\}$$

b1

$$-\frac{1}{2} + g$$

```
(*To verify the expression in the text notice that ...*)
FullSimplify[a2dlim - (a2d - (1 - 2 g - a) / 2)]
```

0

```
(* Hence we (re)define ...*)
a1dlim = FullSimplify[a1dlim /. b1 → b1d]
a2dlim = FullSimplify[a2d - (1 - 2 g - a) / 2]

pdlim = a0 + a1dlim qa + a2dlim qa^2
```

$$c + \frac{a\,(-1 + c + 2\,g - c\,g)}{-1 + 2\,g}$$

$$-\frac{1}{2} + g$$

$$a0 + \left(c + \frac{a\,(-1 + c + 2\,g - c\,g)}{-1 + 2\,g}\right) qa + \left(-\frac{1}{2} + g\right) qa^2$$

```
(*Now we define the regions RMLD and RLD*)

ctilde = c /. FullSimplify[Solve[b1d == 1 / 2, c]][[1, 1]]
(*since this states that we have that type x=
 b1d which is marginal with the limit  quantity is equal to the type x=
  1/2 which is marginal with the monopoly quantity*)
ctilde = ctilde /. a → as;

xdtilde =
 x /. FullSimplify[Solve[qm == qdlim, x] /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d}][[1, 1]]

qdtilde = FullSimplify[qdlim /. x -> xdtilde /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d}]

(*notice that when the monopoly does not exist and the lowest part
  is characterized by limit pricing then the lowest buying type is x=b1d*)
```

$$\frac{(-1 + 2\,a)\,(-1 + 2\,g)}{2 + 2\,a\,(-1 + g) - 4\,g}$$

$$\frac{-a\,(1 + c\,(-1 + g) - 2\,g)\,(-1 + g) + (c\,(-1 + g) + g)\,(-1 + 2\,g)}{1 + g\,(-5 + 6\,g)}$$

$$\frac{2\,a\,c}{1 - 2\,g} + \frac{(-1 + 2\,a)\,(-1 + 2\,c)}{-1 + 3\,g}$$

```
xdhat =
 x /. FullSimplify[Solve[qbd == 0, x] /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d}][[1, 1]]

qdhat = FullSimplify[qad /. x -> xdhat /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d}]

(*notice the following ...*)
FullSimplify[Solve[qdlim == qad, x] /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d}][[1, 1]]
FullSimplify[xdhat - x /. %]
(*i.e. clearly when qbd is nil at that point A switches to the limit pricing*)
```

$$\frac{c\,(-1+g)+a\,(-1+c+2\,g-c\,g)}{-1+2\,g}$$

$$\frac{c}{1-2\,g}$$

$$x\to\frac{c\,(-1+g)+a\,(-1+c+2\,g-c\,g)}{-1+2\,g}$$

0

```
(*Now we calculate the fix fees and the payoffs in region RMLD*)
a0dlimRMLD = a0 /. FullSimplify[Solve[pm == pdlim /. qa → qdtilde, a0]][[1, 1]]

FullSimplify[Solve[pad == (pdlim /. a0 → a0ddlim) /. qa → qdhat, a0]]
a0dRMLD = a0 /. %[[1, 1]] /. a0ddlim -> a0dlim
```

$$\frac{(-1+c\,(2-4\,g)+2\,a\,(1+c\,(-1+g)-2\,g)+2\,g)^2}{4\,(1-2\,g)^2\,(-1+3\,g)}$$

$$\left\{\left\{a0\to a0ddlim-\frac{c^2\,(-1+a+2\,g)}{2\,(1-2\,g)^2}\right\}\right\}$$

$$a0dlim-\frac{c^2\,(-1+a+2\,g)}{2\,(1-2\,g)^2}$$

```
pm /. qa → qm;
pdlim /. {a1 → a1dlim, a2 -> a2dlim, a0 → a0dlimRMLD} /. qa → qdlim;
pad /. qa -> qad /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /. a0 -> a0dRMLD /.
  a0dlim → a0dlimRMLD;
PraRMLD = FullSimplify[Integrate[%%%, {x, 1/2, xdtilde}] +
   Integrate[%%, {x, xdtilde, xdhat}] + Integrate[%, {x, xdhat, 1}]]

pbd - c qb /. qb → qbd /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /. b0 → 0;
PrbRMLD = FullSimplify[Integrate[%, {x, xdhat, 1}]]

uesc - pm /. qa → qm;
uesc - pdlim /. {a1 → a1dlim, a2 -> a2dlim, a0 → a0dlimRMLD} /. qa → qdlim;
u - pad - pbd /. {qa → qad, qb → qbd} /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /.
    a0 -> a0dRMLD /. a0dlim → a0dlimRMLD /. b0 → 0;
URMLD = FullSimplify[Integrate[%%%, {x, 1/2, xdtilde}] +
   Integrate[%%, {x, xdtilde, xdhat}] + Integrate[%, {x, xdhat, 1}]]

WRMLD = FullSimplify[URMLD + PraRMLD + PrbRMLD]

CarltonRMLD = Max[PrbRMLD, URMLD]
```

$$\frac{1}{12\,(-1+2g)^3\,(-1+3g)}\Big(8a^3\,(1+c\,(-1+g)-2g)^3+4a^2\,(1+c\,(-1+g)-2g)^3\,(-4+3g)-2a\,(1+c\,(-1+g)-2g)^3\,(-5+6g)-(-1+2g)\,\big(-(1-2g)^2+6c\,(1-2g)^2+2c^3\,(-1+g)^2-6c^2\,(1+g\,(-3+2g))\big)\Big)$$

$$\frac{(-1+a)\,a\,(1+c\,(-1+g)-2g)^3}{3\,(-1+2g)^3}$$

$$-\frac{1}{24\,(1-3g)^2\,(-1+2g)^3}\Big(16a^3\,(1+c\,(-1+g)-2g)^3\,(-1+2g)-4a\,(1+c\,(-1+g)-2g)^3\,\big(10-39g+36g^2\big)+4a^2\,(1+c\,(-1+g)-2g)^3\,(11+3g\,(-11+6g))+(-1+2g)\,\big(12c\,(1-2g)^2\,(3+g\,(-11+9g))+4c^3\,(-1+g)^2\,(3+g\,(-11+9g))-12c^2\,(-1+g)\,(-1+2g)\,(3+g\,(-11+9g))-(1-2g)^2\,(13+g\,(-61+72g))\big)\Big)$$

$$\frac{1}{24\,(1-3g)^2\,(-1+2g)^3}\Big(16a^3\,(1+c\,(-1+g)-2g)^3\,g-12a\,(1+c\,(-1+g)-2g)^3\,(-1+2g)+4a^2\,(1+c\,(-1+g)-2g)^3\,(-1+9g\,(-1+2g))-(-1+2g)\,\big(12c\,(1-2g)^2\,(2+g\,(-8+9g))+4c^3\,(-1+g)^2\,(2+g\,(-8+9g))-12c^2\,(-1+g)\,(-1+2g)\,(2+g\,(-8+9g))-(1-2g)^2\,(11+g\,(-55+72g))\big)\Big)$$

$$\mathrm{Max}\Big[\frac{(-1+a)\,a\,(1+c\,(-1+g)-2g)^3}{3\,(-1+2g)^3},\;-\frac{1}{24\,(1-3g)^2\,(-1+2g)^3}\Big(16a^3\,(1+c\,(-1+g)-2g)^3\,(-1+2g)-4a\,(1+c\,(-1+g)-2g)^3\,\big(10-39g+36g^2\big)+4a^2\,(1+c\,(-1+g)-2g)^3\,(11+3g\,(-11+6g))+(-1+2g)\,\big(12c\,(1-2g)^2\,(3+g\,(-11+9g))+4c^3\,(-1+g)^2\,(3+g\,(-11+9g))-12c^2\,(-1+g)\,(-1+2g)\,(3+g\,(-11+9g))-(1-2g)^2\,(13+g\,(-61+72g))\big)\Big)\Big]$$

```
(*Now we calculate the fix fee and the payoffs in region RLD*)
FullSimplify[Solve[pad == (pdlim /. a0 → 0) /. qa → qdhat, a0]]
a0dRLD = a0 /. %[[1, 1]]
```

$$\left\{\left\{a0 \to -\frac{c^2 (-1 + a + 2 g)}{2 (1 - 2 g)^2}\right\}\right\}$$

$$-\frac{c^2 (-1 + a + 2 g)}{2 (1 - 2 g)^2}$$

```
pdlim /. {a1 → a1dlim, a2 -> a2dlim, a0 → 0} /. qa → qdlim;
pad /. qa -> qad /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /. a0 -> a0dRLD;

PraRLD = FullSimplify[Integrate[%%, {x, b1d, xdhat}] + Integrate[%, {x, xdhat, 1}]]


FullSimplify[pbd - c qb /. qb → qbd /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /. b0 → 0];
PrbRLD = FullSimplify[Integrate[%, {x, xdhat, 1}]]


uesc - pdlim /. {a1 → a1dlim, a2 -> a2dlim, a0 → 0} /. qa → qdlim;
u - pad - pbd /. {qa -> qad, qb → qbd} /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /.
  {a0 -> a0dRLD, b0 → 0};
URLD = FullSimplify[Integrate[%%, {x, b1d, xdhat}] + Integrate[%, {x, xdhat, 1}]]

WRLD = FullSimplify[URLD + PraRLD + PrbRLD]
CarltonRLD = Max[PrbRLD, URLD];
```

$$\frac{1}{6 (-1 + 2 g)^3} \left(c^2 g (-2 c + 4 c g + 3 a (-1 + c + 2 g - c g)) + (-1 + a) (1 + c (-1 + g) - 2 g) \left(3 c^2 (1 - 2 g) + 2 a (1 + c (-1 + g) - 2 g) (1 - 2 g + c (2 + g))\right)\right)$$

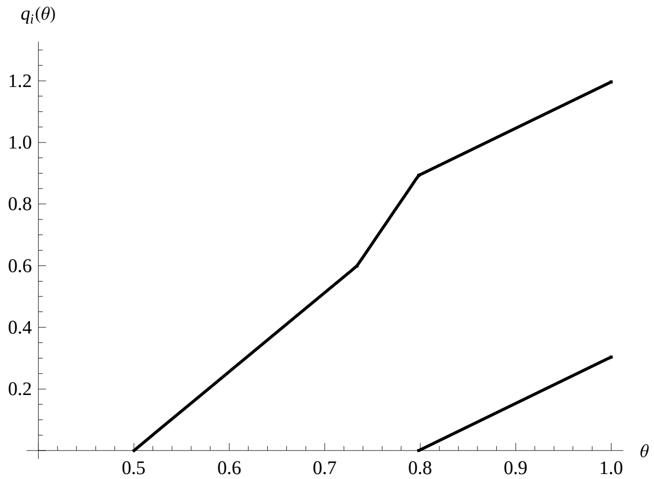
$$\frac{(-1 + a) a (1 + c (-1 + g) - 2 g)^3}{3 (-1 + 2 g)^3}$$

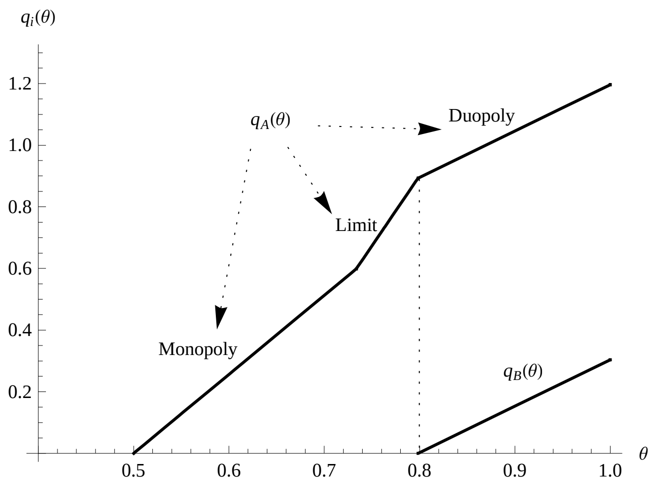
$$-\frac{1}{6 (-1 + 2 g)^3} \left(c^3 g^2 + (-1 + a) (1 + c (-1 + g) - 2 g) \left(-2 + c + c^2 - 2 (-1 + c) (4 + c) g - 2 (-2 + c)^2 g^2 + a (1 + c (-1 + g) - 2 g) (2 + c + 2 (-2 + c) g)\right)\right)$$

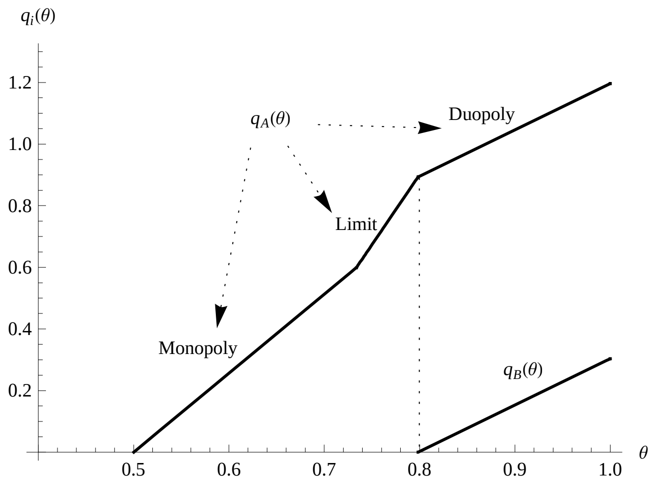
$$\frac{1}{24} \left(\left(-1 + a^2\right) (-2 + c)^3 - \frac{c^2 \left(3 a^2 (-2 + c) + c - 6 a c\right)}{(1 - 2 g)^2} + \frac{2 a^2 c^3}{(-1 + 2 g)^3} - \frac{6 (1 + a (-2 + c) - c) c^2}{-1 + 2 g}\right)$$

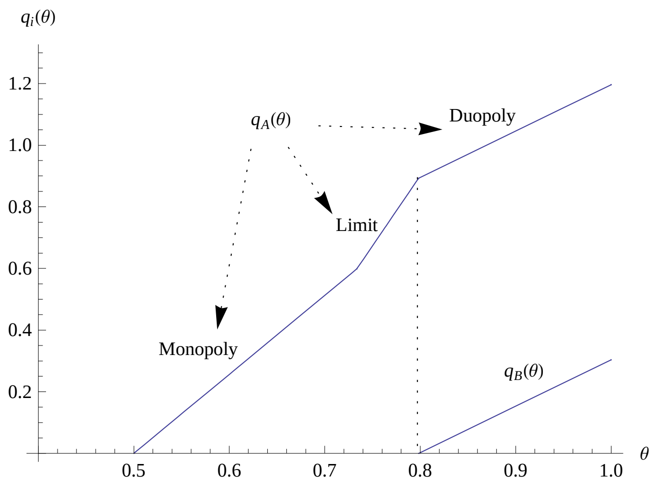
```
(*THIS CONCLUDES THE DERIVATION OF THE EQUILIBRIUM WITH NON-LINEAR PRICING*)


        (* Plotting quantities for non-
 linear pricing: the RMLD region (Figure 1 in the paper)*)
```

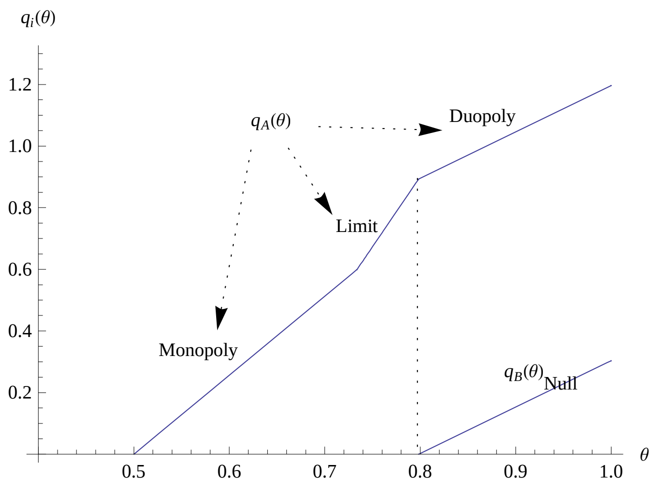
```
parameters = {g → 0.22, c → 1 / 2};
xdhatplot = xdhat /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /. {a → as} /. parameters;
xdtildeitplot = xdtilde /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /. {a → as} /. parameters;

qadplot = Plot[qad /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /. {a → as} /. parameters,
   {x, xdhatplot, 1}, AxesOrigin → {0, 0},
   PlotStyle → {Thickness[0.005], Black}, DisplayFunction → Identity];
qdlimplot = Plot[qdlim /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /. {a → as} /. parameters,
   {x, xdtildeitplot, xdhatplot}, AxesOrigin → {0, 0},
   PlotStyle → {Thickness[0.005], Black}, DisplayFunction → Identity];
qmplot = Plot[qm /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /. {a → as} /. parameters,
   {x, 1 / 2, xdtildeitplot}, AxesOrigin → {0, 0},
   PlotStyle → {Thickness[0.005], Black}, DisplayFunction → Identity];

qbdplot = Plot[qbd /. {a2 → a2d, b2 → b2d} /. {a1 → a1d, b1 → b1d} /. {a → as} /. parameters,
   {x, xdhatplot, 1}, AxesOrigin → {0, 0},
   PlotStyle → {Thickness[0.005], Black}, DisplayFunction → Identity];

Show[qadplot, qdlimplot, qmplot, qbdplot, PlotRange → {0, 1.3}, AxesLabel → {"θ", "qi(θ)"},
 AspectRatio → 0.7, AxesOrigin → {0.4, 0}, DisplayFunction → $DisplayFunction]
```




$q_i(\theta)$
1.2
1.0
0.8
0.6
0.4
0.2
$q_A(\theta)$
Duopoly
Limit
Monopoly
$q_B(\theta)$
0.5
0.6
0.7
0.8
0.9
1.0
θ

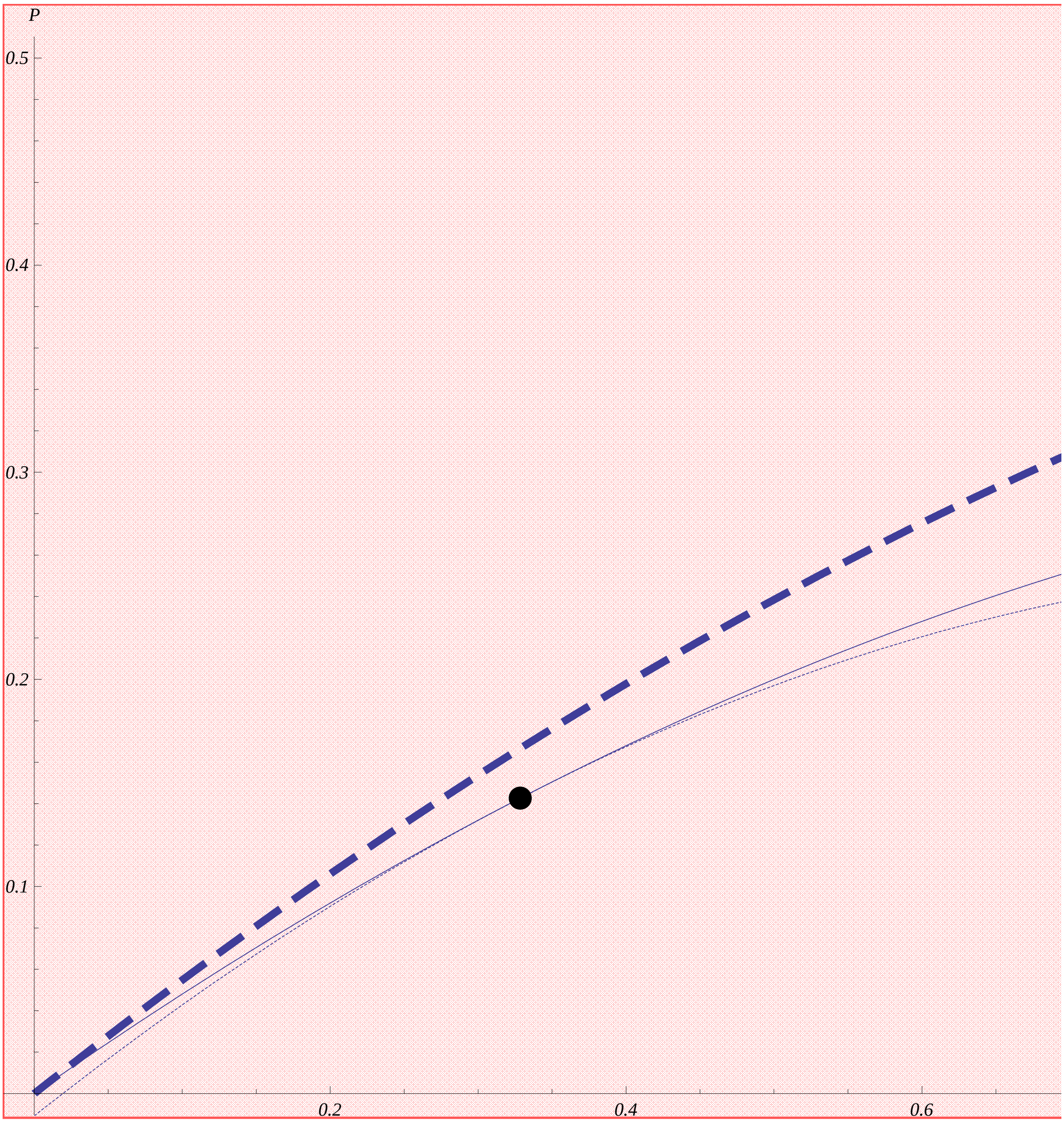

$q_i(\theta)$
1.2
1.0
0.8
0.6
0.4
0.2
$q_A(\theta)$
Duopoly
Limit
Monopoly
$q_B(\theta)$
0.5
0.6
0.7
0.8
0.9
1.0
θ





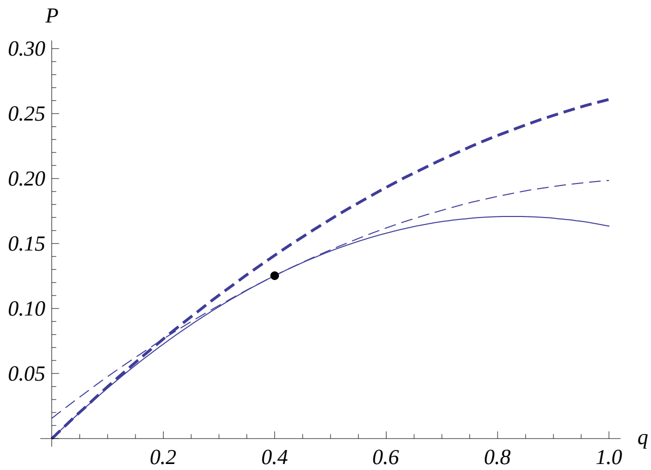

```
(* RMLD plot of the price schedules *)
```

```
parameters = {c → 0.4, g → 0.2};
pmplotRMLD = pm /. parameters /. qa → q;
pdlimplotRMLD = pdlim /. a0 → a0dlimRMLD /. a → as /. parameters /. qa → q;
padplotRMLD = pad /. a0 → a0dRMLD /. a → as /. parameters /. qa → q;
pbduoplotRMLD = pbd /. b0 → 0 /. a → as /. parameters /. qb → q;

qdtildeplot = qdtilde /. a → as /. parameters;
pricedlimconstplot =
  pdlim /. a0 → a0dlimRMLD /. a0lim -> a0limRMLD /. a → as /. parameters /. qa → q /.
   q -> qdtildeplot;

qdhatplot = qdhat /. a → as /. parameters; (*att*)
pricedconstplot =
  pad /. a0 → a0dRMLD /. a0lim -> a0limRMLD /. a → as /. parameters /. qa → q /. q -> qdhatplot;


pbduoplotRMLDshow = Plot[pbduoplotRMLD, {q, 0, 1},
   PlotStyle → {Dashing[{0.02, 0.01}], Thickness[0.005]}, DisplayFunction → Identity];
paduoplotRMLDshow = Plot[padplotRMLD, {q, 0, 1},
   PlotStyle → {Dashing[{0.02, 0.01}]}, DisplayFunction → Identity];
palimplotRMLDshow = Plot[pdlimplotRMLD, {q, 0, 1},
   PlotStyle → {Dashing[{0.002, 0.001}]}, DisplayFunction → Identity];
pointupRMLD = Graphics[{PointSize[0.015], Point[{qdhatplot, pricedconstplot}]}];

pmplotRMLDshow = Plot[pmplotRMLD, {q, 0, 1}, DisplayFunction → Identity];
pointlowRMLD = Graphics[{PointSize[0.015], Point[{qdtildeplot, pricedlimconstplot}]}];

Show[pbduoplotRMLDshow, paduoplotRMLDshow,
 palimplotRMLDshow, pointupRMLD, pmplotRMLDshow, pointlowRMLD,
 AxesLabel → {"q", "P"}, PlotRange → {0, 0.5},
 TextStyle → {FontSlant → "Italic", FontSize → 10}, Ticks -> {Automatic, Automatic},
 AspectRatio → 0.7, DisplayFunction → $DisplayFunction]
```




```
(*this shows that first the mopolist price is admissible,
then it becomes with a too low marginal price and is substituted
 by the limit price and then the duopoly becomes admissible*)

        (* RLD plot of the price schedules*)
```

```
parameters = {c → 0.2, g → 1 / 4};
pdlimplotRLD = pdlim /. a0 → 0 /. a → as /. parameters /. qa → q;
padplotRLD = pad /. a0 → a0dRLD /. a → as /. parameters /. qa → q;
pbdplotRLD = pbd /. b0 → 0 /. a → as /. parameters /. qb → q;
qdhatplotRLD = qdhat /. a → as /. parameters;
priceplotRLD = pdlim /. a0 → 0 /. a → as /. parameters /. qa → q /. q -> qdhatplotRLD;

pbdplotRLDshow = Plot[pbdplotRLD, {q, 0, 1},
   PlotStyle → {Dashing[{0.02, 0.01}], Thickness[0.005]}, DisplayFunction → Identity];
padplotRLDshow = Plot[padplotRLD, {q, 0, 1}, PlotStyle → {Dashing[{0.02, 0.01}]},
   DisplayFunction → Identity];
pdlimplotRLDshow = Plot[pdlimplotRLD, {q, 0, 1},
   PlotStyle → {Dashing[{0.002, 0.001}]}, DisplayFunction → Identity];
pointplotRLD = Graphics[{PointSize[0.015], Point[{qdhatplotRLD, priceplotRLD}]}];

Show[pbdplotRLDshow, padplotRLDshow, pdlimplotRLDshow, pointplotRLD,
 AxesLabel → {"q", "P"}, PlotRange → {0, 0.3},
 TextStyle → {FontSlant → "Italic", FontSize → 10}, Ticks -> {Automatic, Automatic},
 AspectRatio → 0.7, DisplayFunction → $DisplayFunction]
```




```
(*this shows that the limit price has a lower marginal price than
 the duopoly for low quantities and viceversa for high quantities*)

(*PART II EQUILIBRIUM WITH LINEAR PRICING*)
```

```
(*First assume firm B is active*)
tmp = FullSimplify[
  Solve[{D[u - pal qa - pbl qb, qa] == 0, D[u - pal qa - pbl qb, qb] == 0}, {qa, qb}]]
qal = qa /. tmp[[1, 1]]
qbl = qb /. tmp[[1, 2]]

xl = x /. Simplify[Solve[qbl == 0, x]][[1, 1]]
  (* types smaller than this do not purchase from B*)
```

$$\left\{\left\{qa \to \frac{pal - g\,pal - g\,pbl}{-1 + 2\,g} + x,\ qb \to \frac{-pbl + g\,(pal + pbl - 2\,x) + x}{1 - 2\,g}\right\}\right\}$$

$$\frac{pal - g\,pal - g\,pbl}{-1 + 2\,g} + x$$

$$\frac{-pbl + g\,(pal + pbl - 2\,x) + x}{1 - 2\,g}$$

$$\frac{-pbl + g\,(pal + pbl)}{-1 + 2\,g}$$

```
(*Deriving qa for types purchasing only
 from A but when higher types also purchase from B*)
uesc = u /. qb → 0;
qaescl = qa /. FullSimplify[Solve[D[uesc - pal qa, qa] == 0, qa]][[1, 1]]
(*Notice that types lower than pal do not buy at all*)
```

$$\frac{pal - x}{-1 + g}$$

```
(*Calculating total quantities and profits *)
Qa = FullSimplify[   Integrate[qaescl, {x, pal, xl}]  + Integrate[qal, {x, xl, 1}]]
Qb = FullSimplify[Integrate[qbl, {x, xl, 1}]]
Pral = FullSimplify[Qa pal]
Prbl = FullSimplify[Qb (pbl - c)]
```

$$\frac{(-1 + pal)^2 + g^2\,(-2 + pal + pbl)^2 - g\,(4 + 3\,(-2 + pal)\,pal + (-2 + pbl)\,pbl)}{2\,(1 - 2\,g)^2}$$

$$\frac{(1 - pbl + g\,(-2 + pal + pbl))^2}{2\,(1 - 2\,g)^2}$$

$$\frac{pal\,\left((-1 + pal)^2 + g^2\,(-2 + pal + pbl)^2 - g\,(4 + 3\,(-2 + pal)\,pal + (-2 + pbl)\,pbl)\right)}{2\,(1 - 2\,g)^2}$$

$$\frac{(-c + pbl)\,(1 - pbl + g\,(-2 + pal + pbl))^2}{2\,(1 - 2\,g)^2}$$

```
(*Now we derive the equilibrium prices*)

(*consider first the necessary condition for pb*)
FullSimplify[D[Prbl, pbl]]
```

$$\frac{(1 - pbl + g\,(-2 + pal + pbl))\,(1 - 2\,c\,(-1 + g) - 3\,pbl + g\,(-2 + pal + 3\,pbl))}{2\,(1 - 2\,g)^2}$$

```
(*this becames zero for two pb that respectively nullify the first
  and the second term in the numerator. However the first root for pb is
  not an optimum since it implies a zero Qb. Hence the candidate optimum
  is the second one which nullifies the second term in the numerator ...*)
tmp = FullSimplify[Solve[(1 - pbl + g (-2 + pal + pbl)) == 0, pbl]]
FullSimplify[Qb /. tmp[[1, 1]]]
```

$$\left\{\left\{pbl \to -\frac{1 + g\,(-2 + pal)}{-1 + g}\right\}\right\}$$

0

```
tmp = FullSimplify[Solve[1 - 2 c (-1 + g) - 3 pbl + g (-2 + pal + 3 pbl) == 0, pbl]]
pbltmp = pbl /. tmp[[1, 1]]
```

$$\left\{\left\{pbl \to \frac{1 - 2\,c\,(-1 + g) + g\,(-2 + pal)}{3 - 3\,g}\right\}\right\}$$

$$\frac{1 - 2\,c\,(-1 + g) + g\,(-2 + pal)}{3 - 3\,g}$$

```
(*Now we consider the optimal pa*)
tmp = FullSimplify[Solve[D[Pral, pal] == 0, pal]]
pal1 = pal /. tmp[[1, 1]]
pal2 = pal /. tmp[[2, 1]]
```

$$\left\{\left\{pal \to -\frac{1}{3\,(1 + (-3 + g)\,g)}\left(-2 + 2\,g\,(3 + g\,(-2 + pbl)) + \sqrt{\left(1 + g^4\,(-2 + pbl)^2 + 6\,g^3\,pbl\,(-3 + 2\,pbl) + g^2\,(1 + 2\,(11 - 6\,pbl)\,pbl) + 3\,g\,(-1 + (-2 + pbl)\,pbl)\right)}\right)\right\},$$
$$\left\{pal \to \frac{1}{3\,(1 + (-3 + g)\,g)}\left(2 - 2\,g\,(3 + g\,(-2 + pbl)) + \sqrt{1 + g^4\,(-2 + pbl)^2 + 6\,g^3\,pbl\,(-3 + 2\,pbl) + g^2\,(1 + 2\,(11 - 6\,pbl)\,pbl) + 3\,g\,(-1 + (-2 + pbl)\,pbl)}\right)\right\}\right\}$$

$$-\frac{1}{3\,(1 + (-3 + g)\,g)}\left(-2 + 2\,g\,(3 + g\,(-2 + pbl)) + \sqrt{1 + g^4\,(-2 + pbl)^2 + 6\,g^3\,pbl\,(-3 + 2\,pbl) + g^2\,(1 + 2\,(11 - 6\,pbl)\,pbl) + 3\,g\,(-1 + (-2 + pbl)\,pbl)}\right)$$

$$\frac{1}{3\,(1 + (-3 + g)\,g)}\left(2 - 2\,g\,(3 + g\,(-2 + pbl)) + \sqrt{1 + g^4\,(-2 + pbl)^2 + 6\,g^3\,pbl\,(-3 + 2\,pbl) + g^2\,(1 + 2\,(11 - 6\,pbl)\,pbl) + 3\,g\,(-1 + (-2 + pbl)\,pbl)}\right)$$

```
FullSimplify[pal1 - pal2]
```

$$-\frac{2\sqrt{1 + g^4\,(-2 + pbl)^2 + 6\,g^3\,pbl\,(-3 + 2\,pbl) + g^2\,(1 + 2\,(11 - 6\,pbl)\,pbl) + 3\,g\,(-1 + (-2 + pbl)\,pbl)}}{3\,(1 + (-3 + g)\,g)}$$

```
palsola = pal /. FullSimplify[Solve[pal == pal1 /. pbl -> pbltmp, pal]][[1, 1]]
palsolb = pal /. FullSimplify[Solve[pal == pal1 /. pbl -> pbltmp, pal]][[2, 1]]
tmp = FullSimplify[Solve[pal == pal2 /. pbl -> pbltmp, pal]]
```

$$\left(2 g (36 + g (-41 + 5 c - 5 (-2 + c) g)) - 3 \left(6 + \sqrt{\left(9 + 4 g \left((-3 + 2 g) (1 + (-1 + g) g) (5 + g (-5 + 2 g)) - 2 c (-1 + g) (-1 + 2 g) (3 + g (-7 + g (2 + g)))\right) + c^2 (-1 + g)^2 (3 + g (-12 + g (12 + g)))\right)}\right)\right) / ((-3 + 4 g) (9 + 4 (-6 + g) g))$$

$$\left(2 g (36 + g (-41 + 5 c - 5 (-2 + c) g)) + 3 \left(-6 + \sqrt{\left(9 + 4 g \left((-3 + 2 g) (1 + (-1 + g) g) (5 + g (-5 + 2 g)) - 2 c (-1 + g) (-1 + 2 g) (3 + g (-7 + g (2 + g)))\right) + c^2 (-1 + g)^2 (3 + g (-12 + g (12 + g)))\right)}\right)\right) / ((-3 + 4 g) (9 + 4 (-6 + g) g))$$

$$\left\{\left\{pal \to \frac{1}{(-3 + 4 g) (9 + 4 (-6 + g) g)} \left(2 g (36 + g (-41 + 5 c - 5 (-2 + c) g)) - 3 \left(6 + \sqrt{\left(9 + 4 g \left((-3 + 2 g) (1 + (-1 + g) g) (5 + g (-5 + 2 g)) - 2 c (-1 + g) (-1 + 2 g) (3 + g (-7 + g (2 + g)))\right) + c^2 (-1 + g)^2 (3 + g (-12 + g (12 + g)))\right)}\right)\right)\right\}, \left\{pal \to \frac{1}{(-3 + 4 g) (9 + 4 (-6 + g) g)} \left(2 g (36 + g (-41 + 5 c - 5 (-2 + c) g)) + 3 \left(-6 + \sqrt{\left(9 + 4 g \left((-3 + 2 g) (1 + (-1 + g) g) (5 + g (-5 + 2 g)) - 2 c (-1 + g) (-1 + 2 g) (3 + g (-7 + g (2 + g)))\right) + c^2 (-1 + g)^2 (3 + g (-12 + g (12 + g)))\right)}\right)\right)\right\}\right\}$$

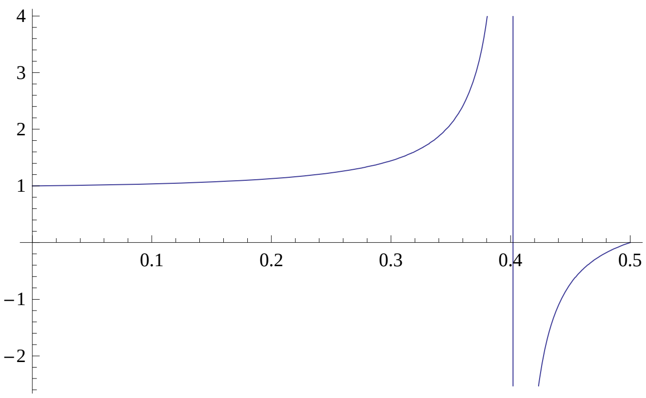
```
FullSimplify[palsola - pal /. tmp[[1, 1]]]
FullSimplify[palsolb - pal /. tmp[[2, 1]]]
```

0

0

```
(*there are two coincident solutions,
hence we take only the first two: palsola palsolb*)

FullSimplify[palsola /. c → 0]
Plot[%, {g, 0, 1/2}]
```

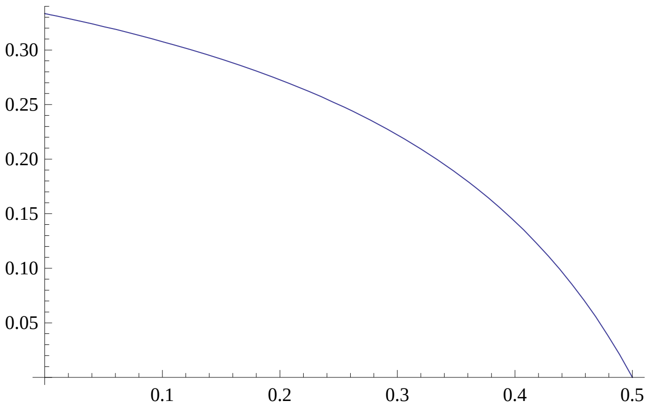
$$\frac{2 g (36 + g (-41 + 10 g)) - 3 \left(6 + \sqrt{(3 - 2 g (5 + g (-5 + 2 g)))^2}\right)}{(-3 + 4 g) (9 + 4 (-6 + g) g)}$$




```
(*this graph shows that the solution palsola is not admissible since either the price
 is larger than one or it is negative and in any case it is increasing in g*)

(*this graph shows that the solution is indeed palsolb *)
```

```
FullSimplify[palsolb /. c → 0]
Plot[%, {g, 0, 1/2}]
```

$$\frac{2\,g\,(36+g\,(-41+10\,g))+3\left(-6+\sqrt{(3-2\,g\,(5+g\,(-5+2\,g)))^2}\right)}{(-3+4\,g)\,(9+4\,(-6+g)\,g)}$$




```
(*notice that the denominator in pa2sol
   becomes zero for certain values of admissible g....*)
```

```
FullSimplify[Solve[Denominator[palsolb] == 0, g]]
```

$$\left\{\left\{g\to\frac{3}{4}\right\},\ \left\{g\to 3-\frac{3\sqrt{3}}{2}\right\},\ \left\{g\to\frac{3}{2}\left(2+\sqrt{3}\right)\right\}\right\}$$

(*in particular this is the case when g= $\frac{3}{2}\left(2+\sqrt{3}\right)$, however,
looking at the necessary condition that delivers the two solutions,
we can see that for that value of g the coefficient of the quadratic term
for pa vanishes and the equation becomes one of first order. This shows
that the fact that the denominator of palsolb vanishes is not a problem,
and the solution should be seen as the limit of palsolb for g that tends
to $\frac{3}{2}\left(2+\sqrt{3}\right)$ which in fact corresponds to the solution of the polynomial
of degree one when the coefficient of the quadratic term vanishes
(this is always the case for the roots of any polynomial of degree two when the
coefficient of the quadratic term can vanish for some values of the parameters).*)

```
palinear = palsolb
```

$$\Big(2\,g\,(36+g\,(-41+5\,c-5\,(-2+c)\,g))+3\Big(-6+\sqrt{\big(9+4\,g\,\big((-3+2\,g)\,(1+(-1+g)\,g)\,(5+g\,(-5+2\,g))-2\,c\,(-1+g)\,(-1+2\,g)\,(3+g\,(-7+g\,(2+g)))+c^2\,(-1+g)^2\,(3+g\,(-12+g\,(12+g)))\big)\big)}\Big)\Big)\Big/\big((-3+4\,g)\,(9+4\,(-6+g)\,g)\big)$$

```
(*now we can calculate the equilibrium price for B*)
(*To this end we simply solve  the system of foc for the prices and determine the
 solution for pb associate to the solution for pa which is equal to pa2sol*)
tmp = FullSimplify[Solve[{D[Pral, pal] == 0, D[Prbl, pbl] == 0}, {pal, pbl}]]
```

$$\Big\{\{\text{pbl}\to 1,\ \text{pal}\to 1\},\ \Big\{\text{pbl}\to \frac{5}{3}+\frac{2}{3\,(-1+g)},\ \text{pal}\to \frac{1}{3}\Big\},$$

$$\Big\{\text{pbl}\to \frac{1}{(-1+g)\,(-3+4\,g)\,(9+4\,(-6+g)\,g)}\Big(9+2\,c\,(-1+g)\,(-9+g\,(36+g\,(-36+7\,g)))+g\,\Big(-48+2\,(-2+g)\,g\,(-21+2\,g)-\sqrt{\Big(9+4\,g\,\Big((-3+2\,g)\,(1+(-1+g)\,g)\,(5+g\,(-5+2\,g))-2\,c\,(-1+g)\,(-1+2\,g)\,(3+g\,(-7+g\,(2+g)))+c^2\,(-1+g)^2\,(3+g\,(-12+g\,(12+g)))\Big)\Big)}\Big)\Big),$$

$$\text{pal}\to \frac{1}{(-3+4\,g)\,(9+4\,(-6+g)\,g)}\Big(2\,g\,(36+g\,(-41+5\,c-5\,(-2+c)\,g))+3\,\Big(-6+\sqrt{\Big(9+4\,g\,\Big((-3+2\,g)\,(1+(-1+g)\,g)\,(5+g\,(-5+2\,g))-2\,c\,(-1+g)\,(-1+2\,g)\,(3+g\,(-7+g\,(2+g)))+c^2\,(-1+g)^2\,(3+g\,(-12+g\,(12+g)))\Big)\Big)}\Big)\Big)\Big\},$$

$$\Big\{\text{pbl}\to \frac{1}{(-1+g)\,(-3+4\,g)\,(9+4\,(-6+g)\,g)}\Big(9+2\,c\,(-1+g)\,(-9+g\,(36+g\,(-36+7\,g)))+g\,\Big(-48+2\,(-2+g)\,g\,(-21+2\,g)+\sqrt{\Big(9+4\,g\,\Big((-3+2\,g)\,(1+(-1+g)\,g)\,(5+g\,(-5+2\,g))-2\,c\,(-1+g)\,(-1+2\,g)\,(3+g\,(-7+g\,(2+g)))+c^2\,(-1+g)^2\,(3+g\,(-12+g\,(12+g)))\Big)\Big)}\Big)\Big),$$

$$\text{pal}\to \frac{1}{(-3+4\,g)\,(9+4\,(-6+g)\,g)}\Big(2\,g\,(36+g\,(-41+5\,c-5\,(-2+c)\,g))-3\,\Big(6+\sqrt{\Big(9+4\,g\,\Big((-3+2\,g)\,(1+(-1+g)\,g)\,(5+g\,(-5+2\,g))-2\,c\,(-1+g)\,(-1+2\,g)\,(3+g\,(-7+g\,(2+g)))+c^2\,(-1+g)^2\,(3+g\,(-12+g\,(12+g)))\Big)\Big)}\Big)\Big)\Big\}\Big\}$$

```
FullSimplify[palinear - pal /. tmp[[3, 2]]]
pblinear = pbl /. tmp[[3, 1]]
```

0

$$\frac{1}{(-1+g)\,(-3+4\,g)\,(9+4\,(-6+g)\,g)}\Big(9+2\,c\,(-1+g)\,(-9+g\,(36+g\,(-36+7\,g)))+g\,\Big(-48+2\,(-2+g)\,g\,(-21+2\,g)-\sqrt{\Big(9+4\,g\,\Big((-3+2\,g)\,(1+(-1+g)\,g)\,(5+g\,(-5+2\,g))-2\,c\,(-1+g)\,(-1+2\,g)\,(3+g\,(-7+g\,(2+g)))+c^2\,(-1+g)^2\,(3+g\,(-12+g\,(12+g)))\Big)\Big)}\Big)\Big)$$

```
(*Rewriting the equilibrium prices in more compact way, asin the text of the paper*)

g0 = 3 - 2 g (5 + g (-5 + 2 g));
g1 = -8 (-1 + g) g (-1 + 2 g) (3 + g (-7 + g (2 + g)));
g2 = 4 (-1 + g)^2 g (3 + g (-12 + g (12 + g)));
```

$$gc = \sqrt{g0\,^\wedge 2 + g1\, c + g2\, c\,^\wedge 2}\;;$$

```
ga3 = 2 (3 - g) (1 - 2 g) (3 - 5 g);
ga4 = (3 - 4 g) (9 + 4 (-6 + g) g);
```

$$ga5 = \frac{10\,(-1 + g)\, g^2}{ga4};$$

```
gb3 = 9 + 2 g (-24 + (-2 + g) g (-21 + 2 g));
gb4 = (1 - g) ga4;
```

$$gb5 = \frac{2\,(9 - g\,(36 + g\,(-36 + 7\,g)))}{ga4};$$

$$\text{FullSimplify}\left[\text{pblinear} - \left(\frac{gb3 - g\ gc}{gb4} + c\ gb5\right)\right]$$

$$\text{FullSimplify}\left[\text{palinear} - \left(\frac{ga3 - 3\ gc}{ga4} + c\ ga5\right)\right]$$

0

0

```
(*Now we the analysis when B is instead not active*)

FullSimplify[1 - xl /. {pal → palinear, pbl → pblinear}];
tmp = FullSimplify[Solve[% == 0, c]]
cl = c /. tmp[[2, 1]]
```

$$\left\{\{c \to 1\},\ \left\{c \to \frac{5}{3} + \frac{2}{3\,(-1 + g)}\right\}\right\}$$

$$\frac{5}{3} + \frac{2}{3\,(-1 + g)}$$

```
(*when c>cl B is not active, but can it affect A obliging it to limit price?*)

uesc = u /. qb → 0
```

$$-\frac{1}{2}\,(1 - g)\, qa^2 + qa\, x$$

```
(* We first caluclate the optimal monopoly price and check
 when this is admissible as an equilibrium when  B is not active*)
```

```
qmonlinear = qa /. FullSimplify[Solve[D[uesc - pal qa, qa] == 0, qa]][[1, 1]]
Qmonlinear = FullSimplify[Integrate[qmonlinear, {x, pal, 1}]]
tmp = FullSimplify[Solve[D[Qmonlinear pal, pal] == 0, pal]]
pmonlinear = pal /. tmp[[1, 1]]
```

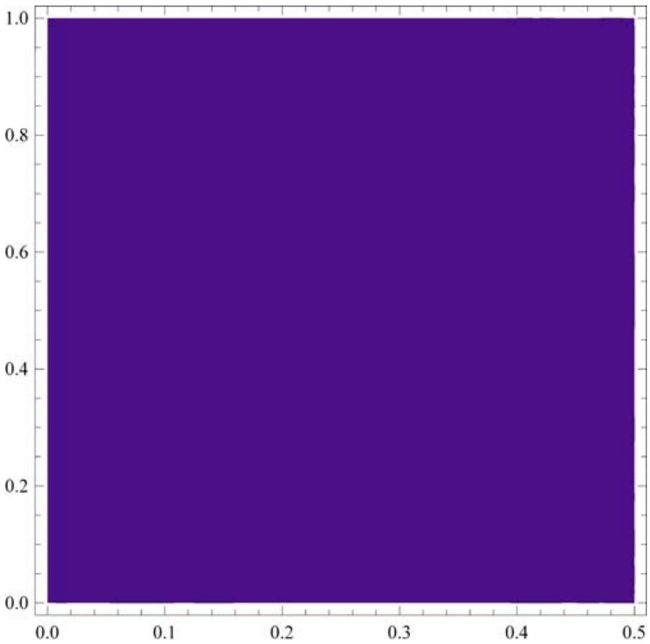
$\frac{\text{pal} - x}{-1 + g}$

$-\frac{(-1 + \text{pal})^2}{2\,(-1 + g)}$

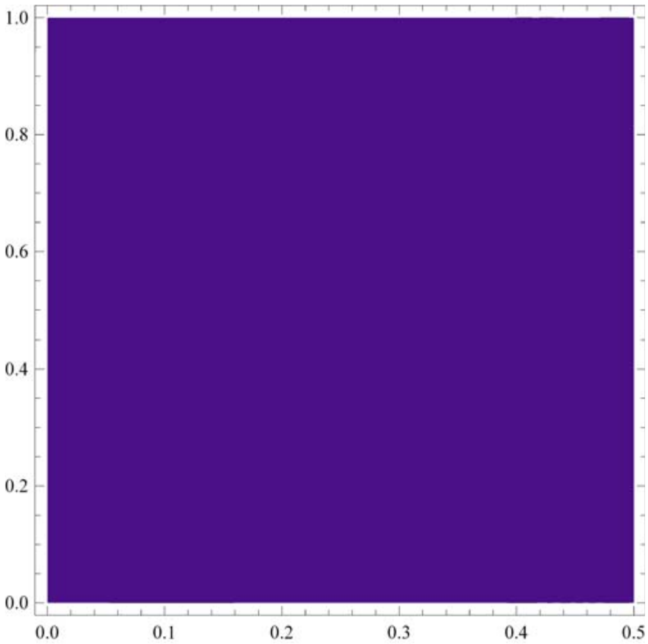
$\left\{\left\{\text{pal} \to \frac{1}{3}\right\}, \{\text{pal} \to 1\}\right\}$

$\frac{1}{3}$

```
(*now, it is clear that for any c>1/3 if A sells as a monopolist,
it does so as uncontested monopolist since B's lowest price=c is larger*)

(*also notice that indeed A makes a lower price than B*)
ContourPlot[palinear - pblinear, {g, 0, 1/2}, {c, 0, 1}, Contours → {0}]
```



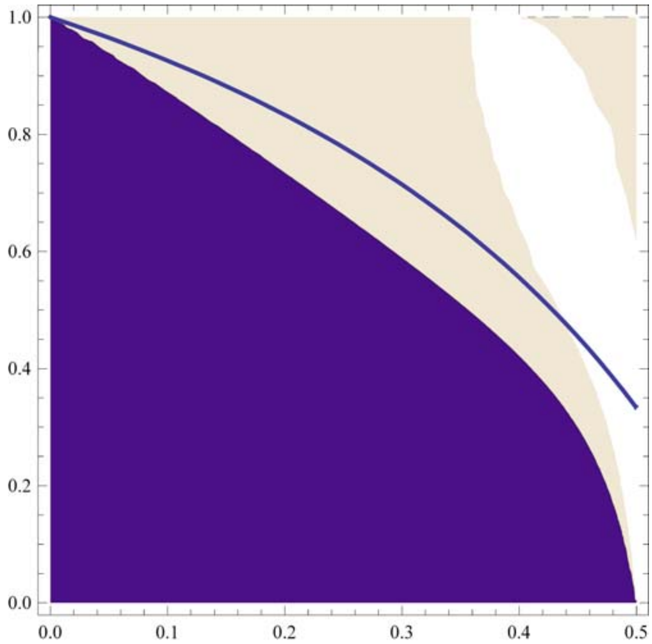
```
(*and notice that A sells to some low types as monopolist with the palinear price*)
xlplot = xl /. {pal → palinear, pbl → pblinear};
ContourPlot[palinear - xlplot, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0}]
```

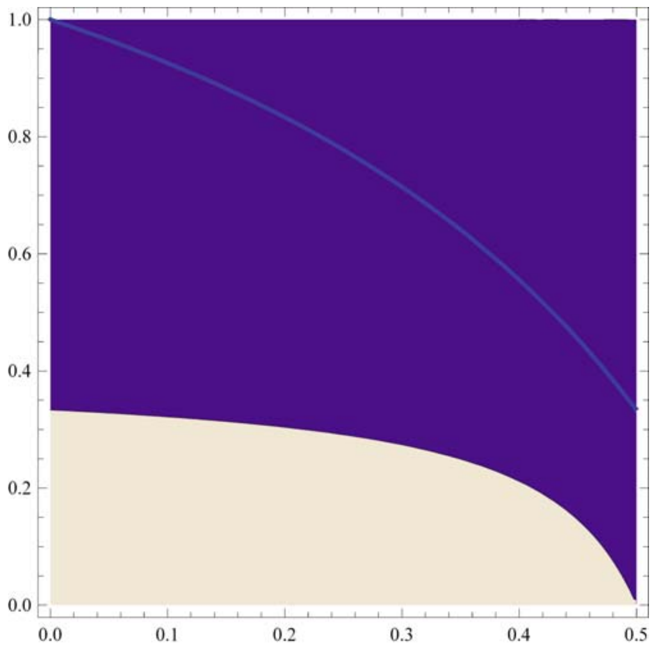



```
(*Some analysis of equilibrium prices*)
```

```
(*linear Price of A*)
pltcl = Plot[cl, {g, 0, 1 / 2}, PlotStyle → {Thick}];

Limit[palinear, g → 1 / 2]
Dpalinear = D[palinear, g];
Dpalinearplot = ContourPlot[Dpalinear, {g, 0, 1 / 2 - 0.000001}, {c, 0, 1}, Contours → {0}];
Show[Dpalinearplot, pltcl]

plotpalminusc = ContourPlot[palinear - c, {g, 0, 1 / 2 - 0.000001}, {c, 0, 1}, Contours → {0}];
Show[plotpalminusc, pltcl]
```

$$\frac{1}{8}\left(5\,c+3\sqrt{c^2}\right)$$

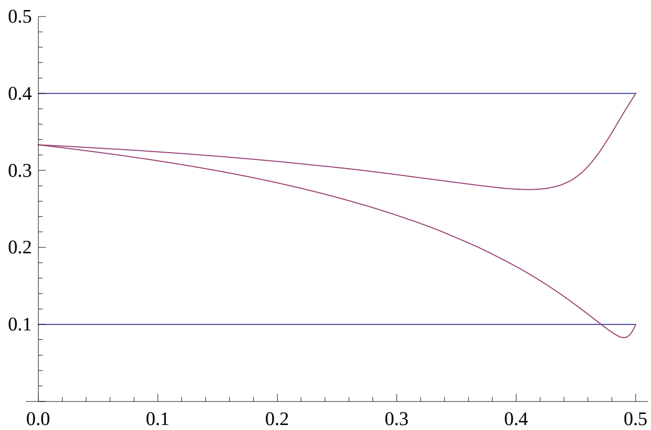




```
Solve[c == palinear /. g → 0, c]
```

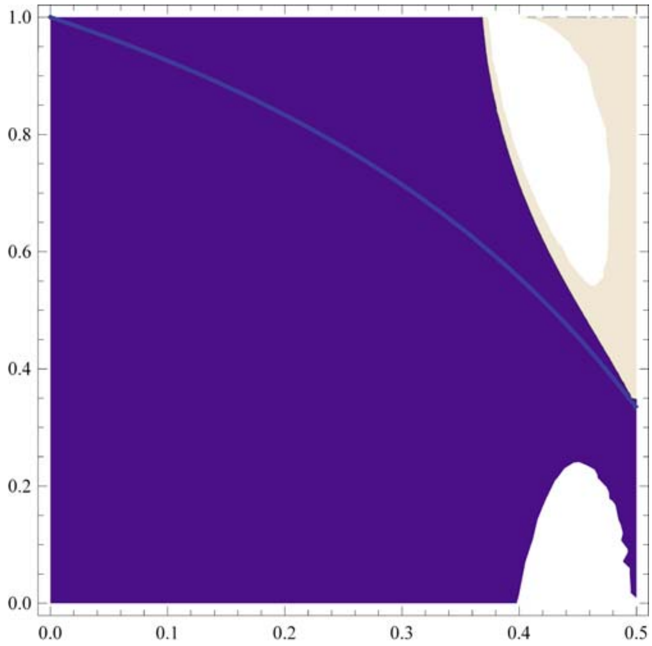
$$\left\{\left\{c \to \frac{1}{3}\right\}\right\}$$

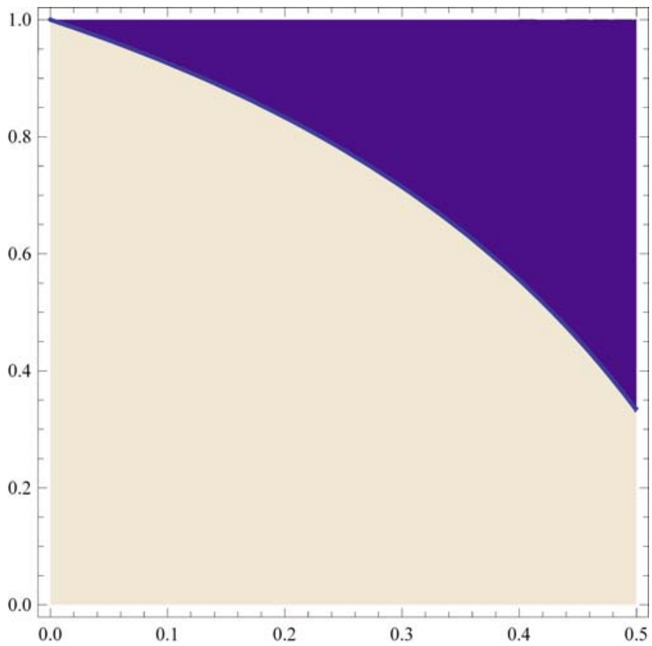
```
cplot = 0.4;
palinearplot = palinear /. c → cplot;
plotafirst = Plot[{cplot, palinearplot}, {g, 0, 1 / 2}, PlotRange → {0, 0.5}];

cplot = 0.1;
palinearplot = palinear /. c → cplot;
plotasecond = Plot[{cplot, palinearplot}, {g, 0, 1 / 2}, PlotRange → {0, 0.5}];

Show[plotafirst, plotasecond ]
```



```
(*linear Price of B*)
Limit[pblinear, g → 1 / 2]
Dpblinear = D[pblinear, g];
Dpblinearplot = ContourPlot[Dpblinear, {g, 0, 1 / 2 - 0.000001}, {c, 0, 1}, Contours → {0}];
Show[Dpblinearplot, pltcl]

plotpblminusc = ContourPlot[pblinear - c, {g, 0, 1 / 2 - 0.000001}, {c, 0, 1}, Contours → {0}];
Show[plotpblminusc, pltcl]
```

$$\frac{1}{8}\left(7\,c+\sqrt{c^2}\right)$$





```
(*NOW WE CAN DEFINE THE
 PAYOFF: FIRST THE CASE IN WHICH B IS ACTIVE AND LINEAR PRICING*)

Qadl = FullSimplify[Integrate[qmonlinear, {x, pal, xl}] + Integrate[qal, {x, xl, 1}]];
Qbdl = FullSimplify[Integrate[qbl, {x, xl, 1}]];
Pradl = Simplify[Qadl pal]
Prbdl = Simplify[Qbdl (pbl - c)]

udl = FullSimplify[u - pal qa - pbl qb /. {qa → qal, qb → qbl}];
umondl = FullSimplify[uesc - pal qa /. {qa → qmonlinear}];
Udl = FullSimplify[Integrate[umondl, {x, pal, xl}] + Integrate[udl, {x, xl, 1}]]

Wdl = FullSimplify[Pradl + Prbdl + Udl]

Carltondl = Max[Prbdl, Udl]
```

$$\frac{\text{pal}\left((-1+\text{pal})^2+g^2(-2+\text{pal}+\text{pbl})^2-g(4+3(-2+\text{pal})\,\text{pal}+(-2+\text{pbl})\,\text{pbl})\right)}{2(1-2g)^2}$$

$$\frac{(-c+\text{pbl})\,(1-\text{pbl}+g(-2+\text{pal}+\text{pbl}))^2}{2(1-2g)^2}$$

$$-\frac{1}{6(1-2g)^2}\Big(\text{pal}\,(3+(-3+\text{pal})\,\text{pal})+g^2(-2+\text{pal}+\text{pbl})^3+(-2+\text{pbl})\,(1+(-1+\text{pbl})\,\text{pbl})+ g\left(8+9\,\text{pal}^2-3\,\text{pal}^3+\text{pbl}\,(-12+(9-2\,\text{pbl})\,\text{pbl})-3\,\text{pal}\,(4+(-2+\text{pbl})\,\text{pbl})\right)\Big)$$

$$\frac{1}{6(1-2g)^2}\Big(2+\text{pal}^2(-3+2\,\text{pal})+2g^2(-2+\text{pal}+\text{pbl})^2(1+\text{pal}+\text{pbl})+\text{pbl}^2(-3+2\,\text{pbl})+ g\left(-8+9\,\text{pal}^2-6\,\text{pal}^3-6\,\text{pal}\,(-1+\text{pbl})\,\text{pbl}+(9-4\,\text{pbl})\,\text{pbl}^2\right)-3c\,(1-\text{pbl}+g(-2+\text{pal}+\text{pbl}))^2\Big)$$

$$\text{Max}\Big[\frac{(-c+\text{pbl})\,(1-\text{pbl}+g(-2+\text{pal}+\text{pbl}))^2}{2(1-2g)^2},\; -\frac{1}{6(1-2g)^2}\Big(\text{pal}\,(3+(-3+\text{pal})\,\text{pal})+g^2(-2+\text{pal}+\text{pbl})^3+(-2+\text{pbl})\,(1+(-1+\text{pbl})\,\text{pbl})+ g\left(8+9\,\text{pal}^2-3\,\text{pal}^3+\text{pbl}\,(-12+(9-2\,\text{pbl})\,\text{pbl})-3\,\text{pal}\,(4+(-2+\text{pbl})\,\text{pbl})\right)\Big)\Big]$$

```
(*WHEN INSTEAD B IS NOT ACTIVE*)

Qamonl = FullSimplify[Integrate[qmonlinear, {x, pal, 1}] /. pal → 1 / 3];
Pramonl = FullSimplify[Qmonlinear pal /. pal → 1 / 3]

umonl = uesc  - pal qa /. qa → qmonlinear /. pal → 1 / 3;
Umonl = FullSimplify[Integrate[umonl, {x, 1 / 3, 1}]]

Wmonl = FullSimplify[Pramonl + Umonl]

Carltonmonl = Max[0, Umonl]
```

$$-\frac{2}{27\,(-1+g)}$$

$$-\frac{4}{81\,(-1+g)}$$

$$-\frac{10}{81\,(-1+g)}$$

$$\text{Max}\left[0,\,-\frac{4}{81\,(-1+g)}\right]$$

```
(*PART III EQUILIBRIUM WITH THE SELECTIVE
 BAN: A CAN ONLY MAKE LINEAR PRICE WHILST B CAN USE LINEAR PRICES*)

(*we derive the b1 and b2 so that for a given pal they implement the quantity that B
 could implement using the optimal direct mechanism with the indirect utility*)

(*calculating the indirect utility for B*)
qatmp = qa /. Solve[D[u - pal qa, qa] == 0, qa][[1, 1]]
vb = FullSimplify[u - pal qa /. qa -> qatmp]
sb = vb - (1 - x) D[vb, x] - c qb (*this is the indirect virtual surplus for B*)
```

$$\frac{\text{pal}+g\,\text{qb}-x}{-1+g}$$

$$\frac{(\text{pal}+\text{qb})\,(\text{pal}+(-1+2\,g)\,\text{qb})-2\,(\text{pal}+(-1+2\,g)\,\text{qb})\,x+x^2}{2-2\,g}$$

$$-c\,\text{qb}-\frac{(1-x)\,(-2\,(\text{pal}+(-1+2\,g)\,\text{qb})+2\,x)}{2-2\,g}+\frac{(\text{pal}+\text{qb})\,(\text{pal}+(-1+2\,g)\,\text{qb})-2\,(\text{pal}+(-1+2\,g)\,\text{qb})\,x+x^2}{2-2\,g}$$

```
(*the the optimal quantity for B is...*)
FullSimplify[Solve[D[sb, qb] == 0, qb]]
```

$$\left\{\left\{\text{qb}\to\frac{1+c-c\,g-g\,(2+\text{pal})}{-1+2\,g}+2\,x\right\}\right\}$$

```
Collect[FullSimplify[Solve[D[vb - pb, qb] == 0, qb]], x]
```

$$\left\{\left\{\text{qb}\to-\frac{\text{b1}\,(-1+g)+g\,\text{pal}}{-1+2\,\text{b2}\,(-1+g)+2\,g}-\frac{(1-2\,g)\,x}{-1+2\,\text{b2}\,(-1+g)+2\,g}\right\}\right\}$$

```
(*now we identify the parameters b1 and be so that the previous qb is
  equal to the optimal one obtained with the direct mechanism approach...*)
```

```
tmp = FullSimplify[Solve[
   {2 == -(1 - 2 g)/(-1 + 2 b2 (-1 + g) + 2 g), (1 + c - c g - g (2 + pal))/(-1 + 2 g) == -(b1 (-1 + g) + g pal)/(-1 + 2 b2 (-1 + g) + 2 g)}, {b1, b2}]]
b1mixtmp = b1 /. tmp[[1, 1]]
b2mix = b2 /. tmp[[1, 2]]
```

$$\left\{\left\{b1 \to \frac{1 + c - c\,g + g\,(-2 + pal)}{2 - 2\,g},\ b2 \to \frac{1 - 2\,g}{4\,(-1 + g)}\right\}\right\}$$

$$\frac{1 + c - c\,g + g\,(-2 + pal)}{2 - 2\,g}$$

$$\frac{1 - 2\,g}{4\,(-1 + g)}$$

```
(*now we find the best pal given these (optimal) b1 and b2 ...*)
```

```
tmp = FullSimplify[Solve[{D[u - pal qa - pb, qb] == 0, D[u - pal qa - pb, qa] == 0}, {qa, qb}]]
qamix = qa /. tmp[[1, 1]]
qbmix = qb /. tmp[[1, 2]]
```

$$\left\{\left\{qa \to \frac{-g\,(b1 + pal - 2\,x) + (1 + 2\,b2)\,(pal - x)}{-1 + 2\,b2\,(-1 + g) + 2\,g},\ qb \to -\frac{b1\,(-1 + g) + g\,(pal - 2\,x) + x}{-1 + 2\,b2\,(-1 + g) + 2\,g}\right\}\right\}$$

$$\frac{-g\,(b1 + pal - 2\,x) + (1 + 2\,b2)\,(pal - x)}{-1 + 2\,b2\,(-1 + g) + 2\,g}$$

$$-\frac{b1\,(-1 + g) + g\,(pal - 2\,x) + x}{-1 + 2\,b2\,(-1 + g) + 2\,g}$$

```
xdmix = x /. FullSimplify[Solve[qbmix == 0, x]][[1, 1]]
```

$$\frac{b1\,(-1 + g) + g\,pal}{-1 + 2\,g}$$

```
Qamonmix = FullSimplify[Integrate[(x - pal) / (1 - g), {x, pal, xdmix}]]
Qadmix = FullSimplify[Integrate[qamix, {x, xdmix, 1}]]
```

$$-\frac{(-1 + g)\,(b1 - pal)^2}{2\,(1 - 2\,g)^2}$$

$$\big((1 + b1\,(-1 + g) + g\,(-2 + pal))\,((1 + 2\,b2)\,(-1 + b1\,(-1 + g) + g\,(2 - 3\,pal) + 2\,pal) + 2\,g\,(1 - pal + g\,(-2 + b1 + pal)))\big) \Big/ \left(2\,(1 - 2\,g)^2\,(-1 + 2\,b2\,(-1 + g) + 2\,g)\right)$$

```
(*then the optimal pal is...*)
palmixtmp = FullSimplify[Solve[D[(Qamonmix + Qadmix) pal, pal] l == 0, pal]]
```

$$\Big\{\Big\{\text{pal} \to \Big(-2 - 4\,b2 + 6\,g + 8\,b2\,g - 4\,g^2 + 2\,b1\,g^2 + \frac{1}{2}\sqrt{\Big(-12\,(1 + b2\,(2 - 4\,g) + (-3 + g)\,g)\,\big(1 + 2\,b2 - (4 - 2\,b1 + b1^2 + 4\,b2)\,g + (-2 + b1)^2\,g^2\big) + 16\,(-1 + b2\,(-2 + 4\,g) + g\,(3 + (-2 + b1)\,g))^2\Big)}\Big) \Big/ (6\,b2\,(-1 + 2\,g) - 3\,(1 + (-3 + g)\,g))\Big\},\ \Big\{\text{pal} \to \Big(-2 - 4\,b2 + 6\,g + 8\,b2\,g - 4\,g^2 + 2\,b1\,g^2 - \frac{1}{2}\sqrt{\Big(-12\,(1 + b2\,(2 - 4\,g) + (-3 + g)\,g)\,\big(1 + 2\,b2 - (4 - 2\,b1 + b1^2 + 4\,b2)\,g + (-2 + b1)^2\,g^2\big) + 16\,(-1 + b2\,(-2 + 4\,g) + g\,(3 + (-2 + b1)\,g))^2\Big)}\Big) \Big/ (6\,b2\,(-1 + 2\,g) - 3\,(1 + (-3 + g)\,g))\Big\}\Big\}$$

```
(*the correct solution is the first one since it is the
 only one to give the right sign to second order condition*)
soca = FullSimplify[D[(Qamonmix + Qadmix) pal, pal, pal]];
FullSimplify[soca /. palmixtmp[[1, 1]] /. b2 -> b2mix]
FullSimplify[soca /. palmixtmp[[2, 1]] /. b2 -> b2mix]
paltmp = pal /. palmixtmp[[1, 1]]
```

$$-\frac{\sqrt{\frac{1+2\,(-1+g)\,g\,\left(1+g\,\left(5-6\,g-4\,g^2+8\,g^3\right)-2\,b1\,(-1+2\,g)\,(3+2\,g\,(-4+g\,(2+g)))+b1^2\,(-1+g)\,(3+2\,g\,(-6+g\,(6+g)))\right)}{(-1+g)^2}}}{(1-2\,g)^2}$$

$$\frac{\sqrt{\frac{1+2\,(-1+g)\,g\,\left(1+g\,\left(5-6\,g-4\,g^2+8\,g^3\right)-2\,b1\,(-1+2\,g)\,(3+2\,g\,(-4+g\,(2+g)))+b1^2\,(-1+g)\,(3+2\,g\,(-6+g\,(6+g)))\right)}{(-1+g)^2}}}{(1-2\,g)^2}$$

$$\Big(-2 - 4\,b2 + 6\,g + 8\,b2\,g - 4\,g^2 + 2\,b1\,g^2 + \frac{1}{2}\sqrt{\Big(-12\,(1 + b2\,(2 - 4\,g) + (-3 + g)\,g)\,\big(1 + 2\,b2 - (4 - 2\,b1 + b1^2 + 4\,b2)\,g + (-2 + b1)^2\,g^2\big) + 16\,(-1 + b2\,(-2 + 4\,g) + g\,(3 + (-2 + b1)\,g))^2\Big)}\Big) \Big/ (6\,b2\,(-1 + 2\,g) - 3\,(1 + (-3 + g)\,g))$$

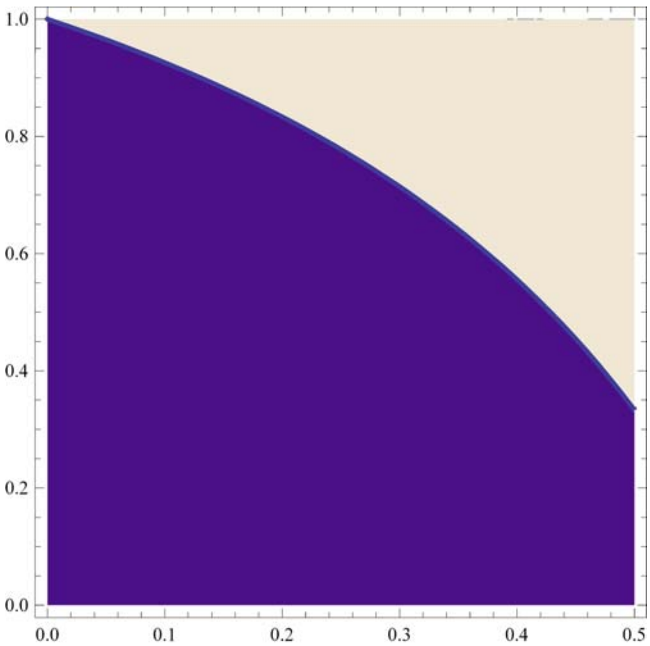
```
tmp = FullSimplify[Solve[pal == paltmp /. b1 -> b1mixtmp /. b2 -> b2mix, pal]]
(*it can be shown that the admissible solution
 is the second (the first it is sometime negative price)*)
palmix = pal /. tmp[[2, 1]]
b1mix = FullSimplify[b1mixtmp /. pal → palmix]
```

$$\left\{\left\{pal \to \left(-4 + g\,(16 + g\,(-19 - 3\,c\,(-1 + g) + 6\,g)) - \sqrt{2}\,\sqrt{\left((-1 + g)\,\left(-2 + g\,\left(-2\,c\,(-1 + 2\,g)\,\left(3 - 6\,g + 2\,g^3\right) + c^2\,(-1 + g)\,(3 + 2\,g\,(-6 + g\,(6 + g))) + (-1 + g)\,(-11 + 2\,g\,(7 + 2\,g\,(-5 + 2\,g)))\right)\right)\right)}\right) \Big/ (-6 + g\,(24 + g\,(-24 + 5\,g)))\right\},\ \left\{pal \to \left(-4 + g\,(16 + g\,(-19 - 3\,c\,(-1 + g) + 6\,g)) + \sqrt{2}\,\sqrt{\left((-1 + g)\,\left(-2 + g\,\left(-2\,c\,(-1 + 2\,g)\,\left(3 - 6\,g + 2\,g^3\right) + c^2\,(-1 + g)\,(3 + 2\,g\,(-6 + g\,(6 + g))) + (-1 + g)\,(-11 + 2\,g\,(7 + 2\,g\,(-5 + 2\,g)))\right)\right)\right)}\right) \Big/ (-6 + g\,(24 + g\,(-24 + 5\,g)))\right\}\right\}$$

$$\left(-4 + g\,(16 + g\,(-19 - 3\,c\,(-1 + g) + 6\,g)) + \sqrt{2}\,\sqrt{\left((-1 + g)\,\left(-2 + g\,\left(-2\,c\,(-1 + 2\,g)\,\left(3 - 6\,g + 2\,g^3\right) + c^2\,(-1 + g)\,(3 + 2\,g\,(-6 + g\,(6 + g))) + (-1 + g)\,(-11 + 2\,g\,(7 + 2\,g\,(-5 + 2\,g)))\right)\right)\right)}\right) \Big/ (-6 + g\,(24 + g\,(-24 + 5\,g)))$$

$$\frac{1}{2 - 2\,g}\left(1 + c - c\,g + \left(g\,\left(8 + g\,(-32 + (29 - 3\,c\,(-1 + g) - 4\,g)\,g) + \sqrt{2}\,\sqrt{\left((-1 + g)\,\left(-2 + g\,\left(-2\,c\,(-1 + 2\,g)\,\left(3 - 6\,g + 2\,g^3\right) + c^2\,(-1 + g)\,(3 + 2\,g\,(-6 + g\,(6 + g))) + (-1 + g)\,(-11 + 2\,g\,(7 + 2\,g\,(-5 + 2\,g)))\right)\right)\right)}\right)\right) \Big/ (-6 + g\,(24 + g\,(-24 + 5\,g)))\right)$$

```
xcritbsol = xdmix /. {pal → palmix, b1 → b1mix};
plotxcritb = ContourPlot[xcritbsol,
   {g, 0, 1 / 2}, {c, 0, 1}, Contours → {1}, DisplayFunction → Identity];
plotclimitl = Plot[cl, {g, 0, 1 / 2}, DisplayFunction → Identity,
   PlotStyle → {Thickness[0.008]}];
Show[plotxcritb, plotclimitl, DisplayFunction → $DisplayFunction]
(*this shows that the two boundaries with linear pricing and
 with mix case are coincident, indeed the last marginal consumer
 for B is the same in the two cases and then its best reponse*)
```



```
Qamonmix = FullSimplify[Integrate[(x - pal) / (1 - g), {x, pal, xdmix}]]
Qadmix = FullSimplify[Integrate[qamix, {x, xdmix, 1}]]
```

$$-\frac{(-1+g)\,(b1-pal)^2}{2\,(1-2g)^2}$$

$$((1+b1\,(-1+g)+g\,(-2+pal))\,((1+2\,b2)\,(-1+b1\,(-1+g)+g\,(2-3\,pal)+2\,pal)+2\,g\,(1-pal+g\,(-2+b1+pal))))\Big/\left(2\,(1-2g)^2\,(-1+2\,b2\,(-1+g)+2g)\right)$$

```
qbmix
```

$$-\frac{b1\,(-1+g)+g\,(pal-2x)+x}{-1+2\,b2\,(-1+g)+2g}$$

```
(*DEFINING THE PAYOFFS WITH B ACTIVE IN THE MIX CASE (SELECTIVE BAN)*)

Pradmix = Simplify[(Qamonmix + Qadmix) * pal];

Prbdmix = Integrate[(b1 - c) qbmix + b2 qbmix^2, {x, xdmix, 1}];

udmix = u - pal qa - b1 qb - b2 qb^2 /. {qa → qamix, qb → qbmix};
umonmix = uesc - pal qa /. {qa → (x - pal) / (1 - g)};
Udmix = Integrate[umonmix, {x, pal, xdmix}] + Integrate[udmix, {x, xdmix, 1}];

Wdmix = Pradmix + Prbdmix + Udmix;
Carltondmix = Max[Prbdmix, Udmix];

(*DEFINING THE PAYOFF IN THE MIX CASE WHEN B IS NOT ACTIVE*)
```

$$pmonmix = \frac{1}{3};$$

$$qmonmix = \frac{pal - x}{-1 + g};$$

```
Qamonmix = FullSimplify[Integrate[qmonmix, {x, pal, 1}] /. pal → 1 / 3];
Pramonmix = FullSimplify[Qamonmix pal /. pal → 1 / 3]

umonmix = uesc  - pal qa /. qa → qmonmix /. pal → 1 / 3;
Umonmix = FullSimplify[Integrate[umonmix, {x, 1 / 3, 1}]]

Wmonmix = FullSimplify[Pramonmix + Umonmix]

Carltonmonmix = Max[0, Umonmix]
```

$$\frac{2}{27-27g}$$

$$-\frac{4}{81\,(-1+g)}$$

$$\frac{10}{81-81g}$$

$$Max\left[0,\ -\frac{4}{81\,(-1+g)}\right]$$

```
(*PART V DERIVING SOME EXPLICIT CASES*)

(*THE SYMMETRIC CASE WITH c=0*)

(*Simplifying the linear prices...*)
FullSimplify[{palinear, pblinear} /. c → 0]
```

$$\left\{\frac{2\,g\,(36+g\,(-41+10\,g))+3\left(-6+\sqrt{(3-2\,g\,(5+g\,(-5+2\,g)))^2}\right)}{(-3+4\,g)\,(9+4\,(-6+g)\,g)},\ \frac{9+g\left(-48+2\,(-2+g)\,g\,(-21+2\,g)-\sqrt{(3-2\,g\,(5+g\,(-5+2\,g)))^2}\right)}{(-1+g)\,(-3+4\,g)\,(9+4\,(-6+g)\,g)}\right\}$$

```
(*For some reasons Mathematica does not the simplification of the sqrt...*)
```

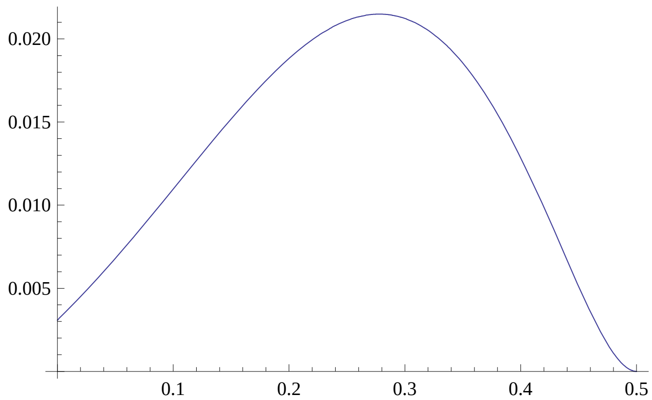
$$\text{FullSimplify}\left[\left\{\frac{2\,g\,(36+g\,(-41+10\,g))+3\,(-6+(3-2\,g\,(5+g\,(-5+2\,g))))}{(-3+4\,g)\,(9+4\,(-6+g)\,g)},\ \frac{9+g\,(-48+2\,(-2+g)\,g\,(-21+2\,g)-(3-2\,g\,(5+g\,(-5+2\,g))))}{(-1+g)\,(-3+4\,g)\,(9+4\,(-6+g)\,g)}\right\}\right]$$

$$\left\{\frac{1-2\,g}{3-4\,g},\ \frac{1-2\,g}{3-4\,g}\right\}$$

$$\text{plcnil}=\frac{1-2\,g}{3-4\,g};$$

```
Wsym = FullSimplify[WRLD /. a → as /. c → 0];
Wsyml = FullSimplify[Wdl /. {pal -> plcnil, pbl -> plcnil} /. c → 0];
FullSimplify[Wsym - Wsyml]
Plot[Wsym - Wsyml, {g, 0, 1 / 2}]
NSolve[Wsym - Wsyml == 0, g]
```

$$\frac{1}{3}\left(1-\frac{4\,(-1+g)^2\,(-5+8\,g)}{(-3+4\,g)^3}-\frac{1}{16}\left(-3+3\,g+\sqrt{1+g\,(-2+9\,g)}\right)^2\right)$$



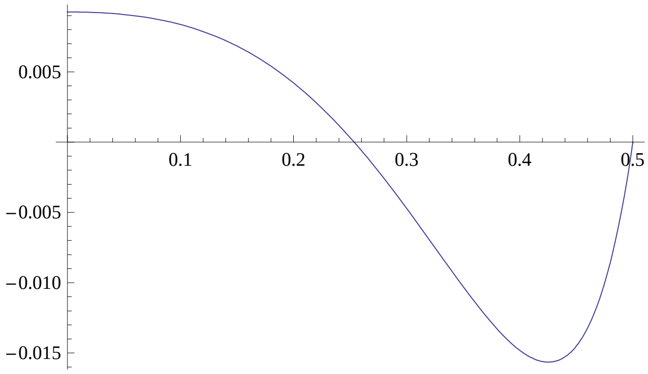
{{g → -1.37571}, {g → 1.16425}, {g → 0.884602}, {g → 0.5}, {g → 0.5}, {g → -0.0506961}}

```
(*firms are symmetric and we then consider only firm A here*)
Prasym = FullSimplify[PraRLD /. a → as /. c → 0]
Prasyml = FullSimplify[Pradl /. {pal -> plcnil, pbl -> plcnil} /. c → 0]
FullSimplify[Prasym - Prasyml]
Plot[Prasym - Prasyml, {g, 0, 1 / 2}]
NSolve[Prasym - Prasyml == 0, g]
```

$$-\frac{1}{48}\left(-3+3\,g+\sqrt{1+g\,(-2+9\,g)}\right)\left(1+3\,g+\sqrt{1+g\,(-2+9\,g)}\right)$$

$$\frac{2\,(-1+g)^2\,(-1+2\,g)}{(-3+4\,g)^3}$$

$$-\frac{2\,(-1+g)^2\,(-1+2\,g)}{(-3+4\,g)^3}-\frac{1}{48}\left(-3+3\,g+\sqrt{1+g\,(-2+9\,g)}\right)\left(1+3\,g+\sqrt{1+g\,(-2+9\,g)}\right)$$



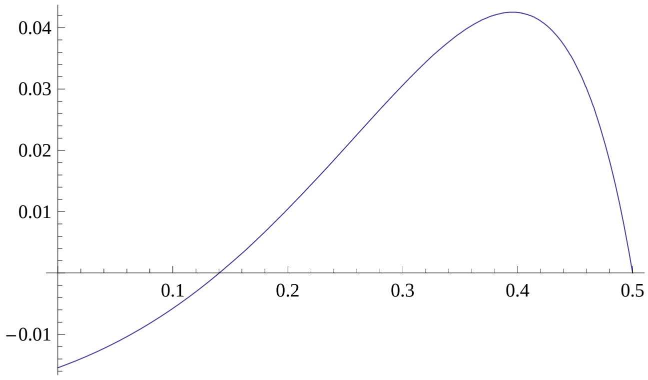
{{g → 0.856129 + 0.0836025 i}, {g → 0.856129 − 0.0836025 i},
{g → 0.103491 + 0.657522 i}, {g → 0.103491 − 0.657522 i}, {g → 0.5}, {g → 0.253523}}

```
Usym = Simplify[URLD /. a → as /. c → 0]
Usyml = Simplify[Udl /. {pal -> plcnil, pbl -> plcnil} /. c → 0]
FullSimplify[Usym - Usyml]
Plot[Usym - Usyml, {g, 0, 1 / 2}]
NSolve[Usym - Usyml == 0, g]
Plot[{Usym, Usyml}, {g, 0, 1 / 2}]
```

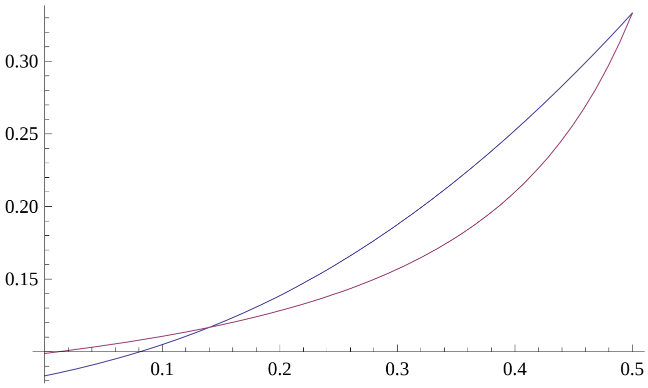
$$\frac{1}{48}\left(1+3\,g+\sqrt{1-2\,g+9\,g^2}\right)^2$$

$$\frac{8\,(-1+g)^3}{3\,(-3+4\,g)^3}$$

$$-\frac{8\,(-1+g)^3}{3\,(-3+4\,g)^3}+\frac{1}{48}\left(1+3\,g+\sqrt{1+g\,(-2+9\,g)}\right)^2$$

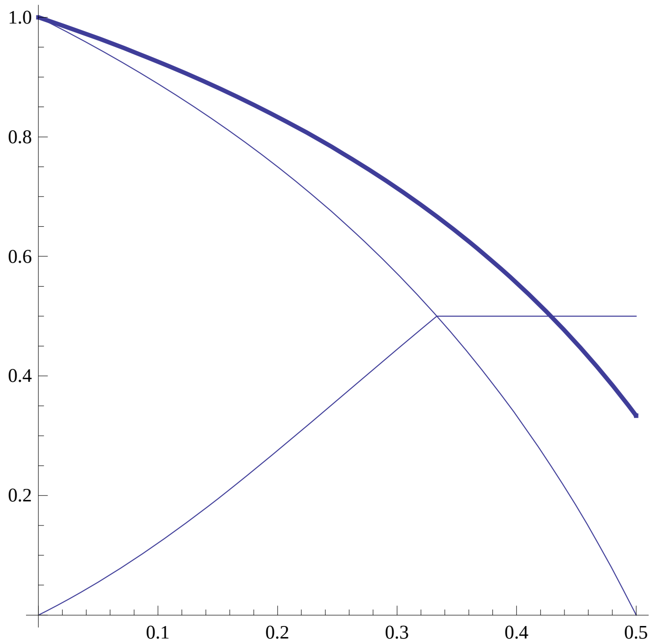


{{g → 0.807843 + 0.0530615 ⅈ}, {g → 0.807843 − 0.0530615 ⅈ}, {g → 0.5}, {g → 0.140376}}

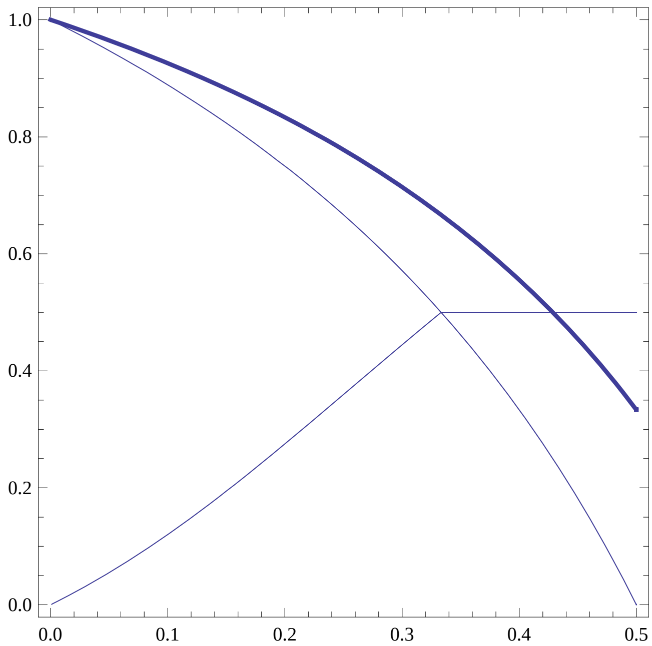


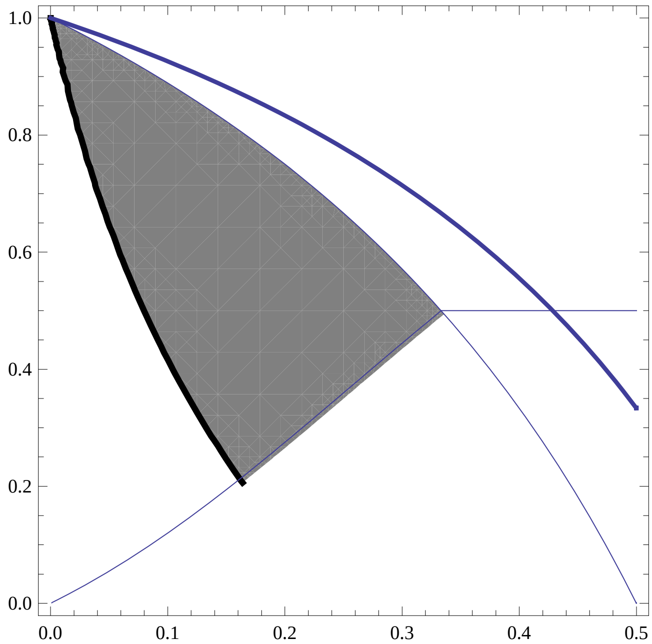

```
(*PART IV PLOTTING THE FIGURES IN THE PAPER*)

    (*COMPARISON NON-LINEAR VERSUS LINEAR*)
```

```
(*Plotting the six regions*)
cdplot = Plot[{cd}, {g, 0, 1 / 2}, AspectRatio → 1];
ctildeplot = Plot[{ctilde /. a → as}, {g, 0.001, 1 / 3}];
clplot = Plot[{cl}, {g, 0, 1 / 2}, PlotStyle → {Thick}];
clplotthick = Plot[{cl}, {g, 0, 1 / 2}, AspectRatio → 1,
   PlotRange → {0, 1}, PlotStyle → {Thickness[0.015], Black}];
constplot = Plot[{1 / 2}, {g, 1 / 3, 1 / 2}];
regionsplot = Show[cdplot, ctildeplot, clplot, constplot]
regionsplotfew = Show[cdplot, ctildeplot, constplot];
regionsplotveryfew = Show[cdplot, ctildeplot];
regionsplotNLP = Show[cdplot];
regionsplotforB = Show[constplot];
```



```
(*REGION 1*)

DPra1 = PraRMLD - Pradl /. {a → as, pal → palinear, pbl → pblinear};
DPrb1 = PrbRMLD - Prbdl /. {a → as, pal → palinear, pbl → pblinear};
DU1 = URMLD - Udl /. {a → as, pal → palinear, pbl → pblinear};
DW1 = WRMLD - Wdl /. {a → as, pal → palinear, pbl → pblinear};
DCarlton1 = Max[DPrb1, DU1] /. {a → as, pal → palinear, pbl → pblinear};

DPraplot1 = ContourPlot[DPra1, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity,
   RegionFunction → Function[{g, c, z}, cd > c > ctilde], ContourStyle → {Thickness[0.01]}];
Show[DPraplot1, regionsplot, DisplayFunction → $DisplayFunction]

DPrbplot1 = ContourPlot[DPrb1, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity, RegionFunction →
    Function[{g, c, z}, cd > c > ctilde - 0.01], ContourStyle → {Thickness[0.01]} ];
Show[DPrbplot1, regionsplot, DisplayFunction → $DisplayFunction]

DUplot1 = ContourPlot[DU1, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity, RegionFunction →
    Function[{g, c, z}, c > Min[ctilde, 1 / 2]], ContourStyle → {Thickness[0.01]} ];
Show[DUplot1, regionsplot, DisplayFunction → $DisplayFunction]

DWplot1 = ContourPlot[DW1, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity,
   RegionFunction → Function[{g, c, z}, cd > c > ctilde], ContourStyle → {Thickness[0.01]} ];
Show[DWplot1, regionsplot, DisplayFunction → $DisplayFunction]

DCarltonplot1 = ContourPlot[DCarlton1, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity, RegionFunction →
    Function[{g, c, z}, c > Min[ctilde, 1 / 2]], ContourStyle → {Thickness[0.01]} ];
Show[DCarltonplot1 , regionsplot, DisplayFunction → $DisplayFunction]
```




1.0
0.8
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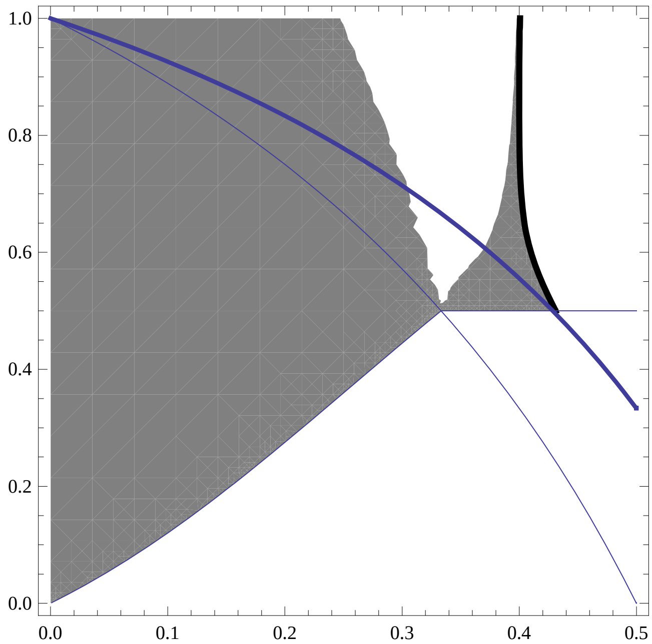
0.5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0.4
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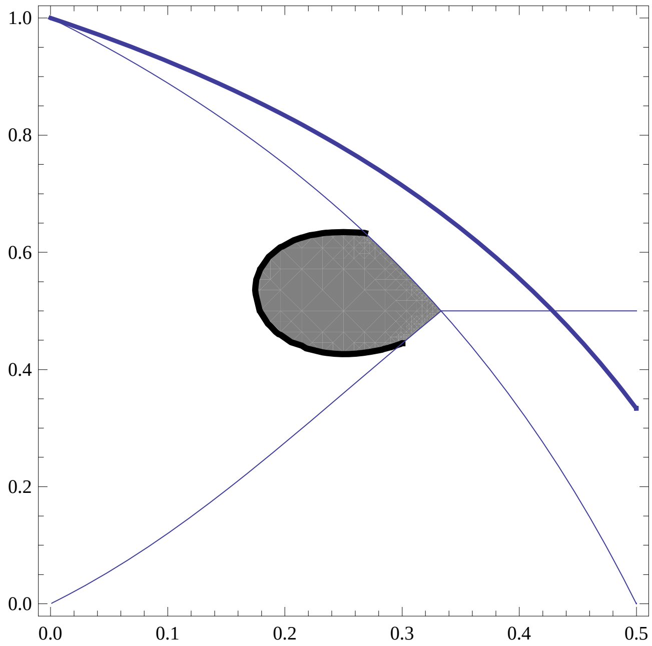
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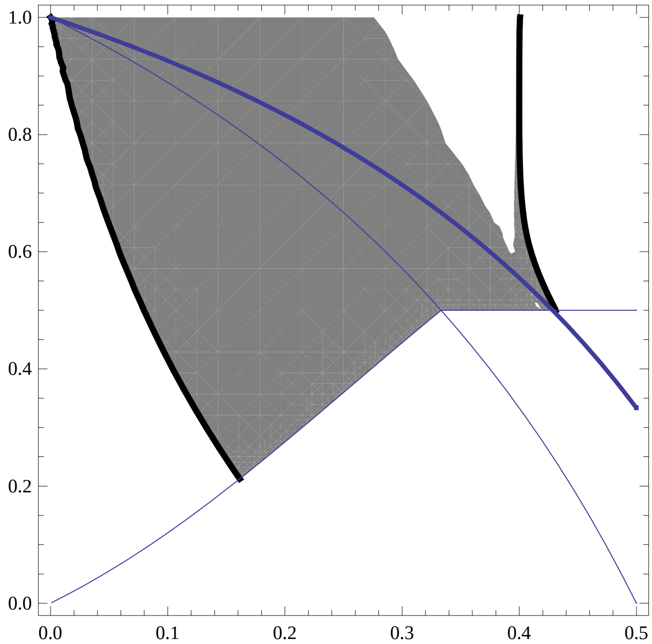
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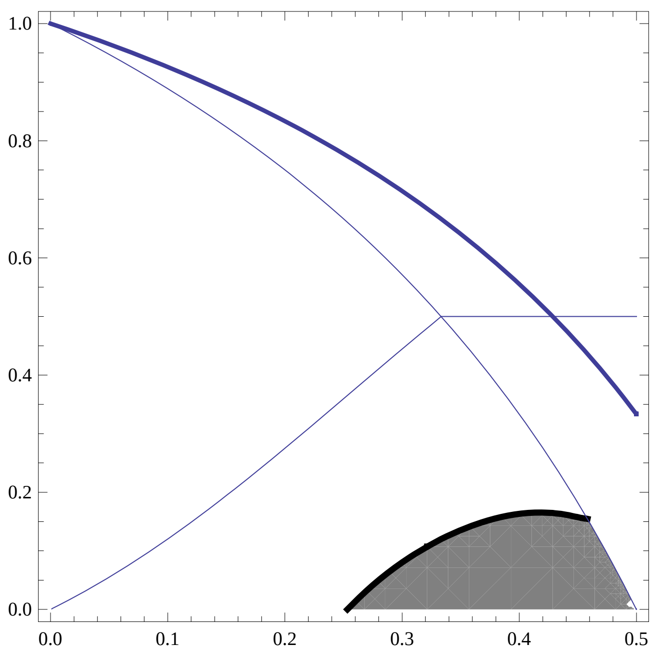
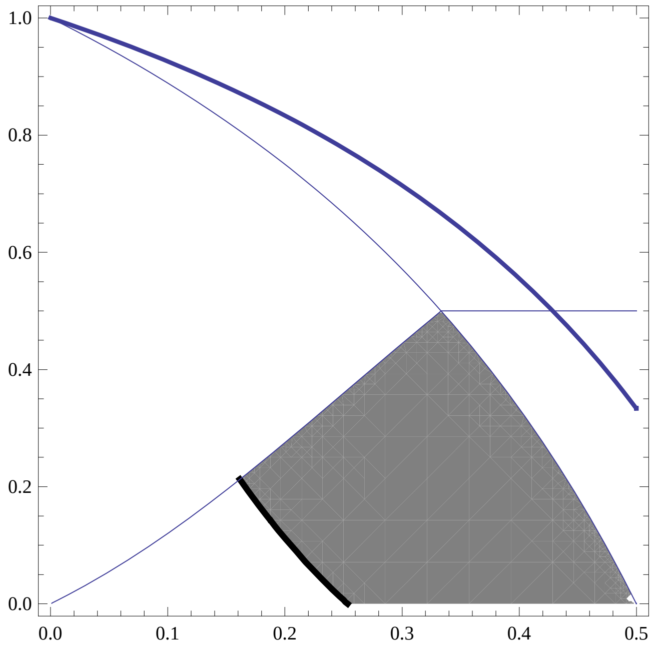
```
(*REGION 2*)

DPra2 = PraRLD - Pradl /. {a → as, pal → palinear, pbl → pblinear};
DPrb2 = PrbRLD - Prbdl /. {a → as, pal → palinear, pbl → pblinear};
DU2 = URLD - Udl /. {a → as, pal → palinear, pbl → pblinear};
DW2 = WRLD - Wdl /. {a → as, pal → palinear, pbl → pblinear};


DPraplot2 = ContourPlot[DPra2, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity, RegionFunction →
    Function[{g, c, z}, Min[cd, ctilde] > c], ContourStyle → {Thickness[0.01]}];
Show[DPraplot2, regionsplot, DisplayFunction → $DisplayFunction]

DPrbplot2 = ContourPlot[DPrb2, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity , RegionFunction →
    Function[{g, c, z}, Min[cd, ctilde] > c], ContourStyle → {Thickness[0.01]}];
Show[DPrbplot2, regionsplot, DisplayFunction → $DisplayFunction]

DUplot2 = ContourPlot[DU2, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity,
   RegionFunction → Function[{g, c, z}, cd > c], ContourStyle → {Thickness[0.01]} ];
Show[DUplot2, regionsplot, DisplayFunction → $DisplayFunction]

DWplot2 = ContourPlot[DW2, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity, RegionFunction →
    Function[{g, c, z}, Min[cd, ctilde] > c], ContourStyle → {Thickness[0.01]} ];
Show[DWplot2, regionsplot, DisplayFunction → $DisplayFunction]

DCarlton2 = Max[DPrb2, DU2] /. {a → as, pa → pal, pb → pbl};
DCarltonplot2 = ContourPlot[DCarlton2, {g, 0.00001, 1 / 2 - 0.00001},
   {c, 0, 1}, Contours → {0}, ColorFunction → (If[#1 > 0, White, Gray] &),
   DisplayFunction → Identity , RegionFunction → Function[{g, c, z}, Min[cd, ctilde] > c],
   ContourStyle → {Thickness[0.01]} ];
Show[DCarltonplot2, regionsplot, DisplayFunction → $DisplayFunction]
```

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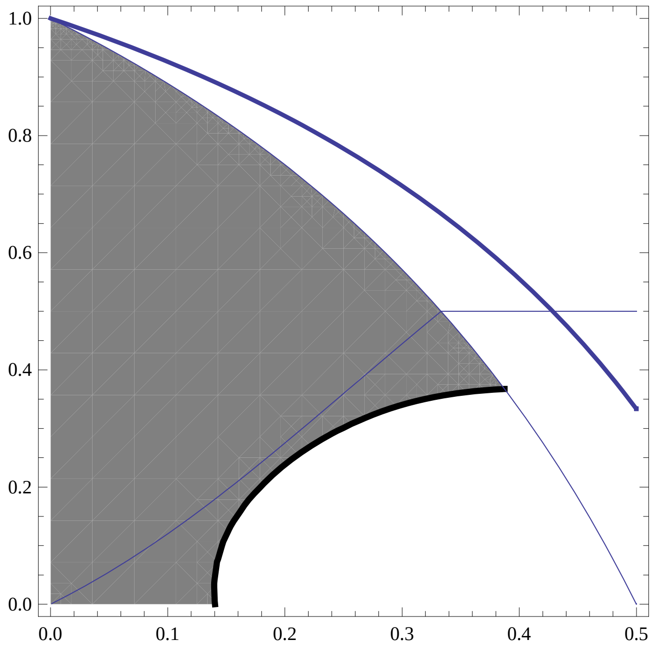
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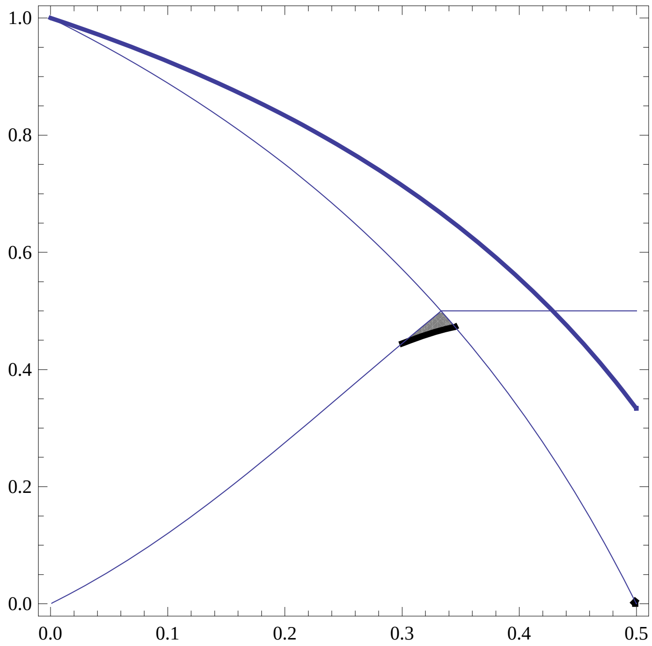
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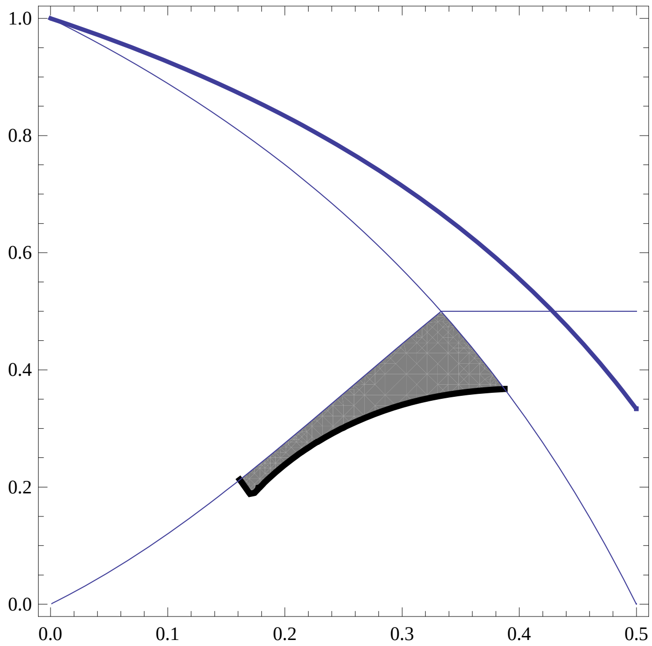
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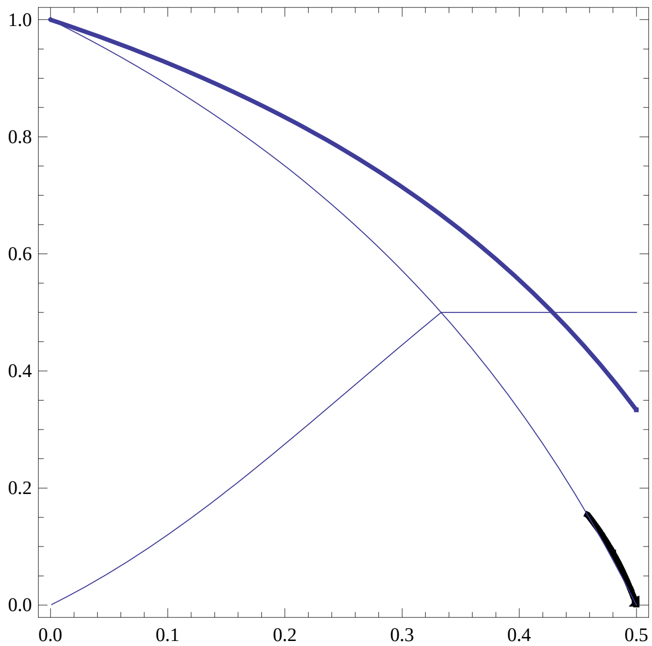
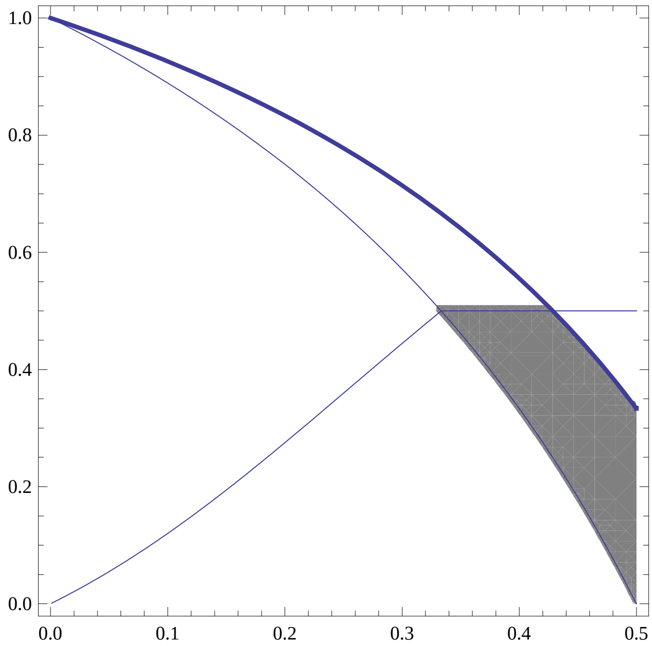
```
(*REGION 3*)

DPra3 = PraRLM - Pradl /. {a → as, pal → palinear, pbl → pblinear};
DPrb3 = PrbRLM - Prbdl /. {a → as, pal → palinear, pbl → pblinear};
DU3 = URLM - Udl /. {a → as, pal → palinear, pbl → pblinear};
DW3 = WRLM - Wdl /. {a → as, pal → palinear, pbl → pblinear};

DPraplot3 = ContourPlot[DPra3, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity , RegionFunction →
    Function[{g, c, z}, Min[1 / 2, cl] > c > cd], ContourStyle → {Thickness[0.01]} ];
Show[DPraplot3, regionsplot, DisplayFunction → $DisplayFunction]

DPrbplot3 = ContourPlot[DPrb3, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity , RegionFunction →
    Function[{g, c, z}, Min[1 / 2, cl] + 0.01 > c > cd - 0.01], ContourStyle → {None} ];
Show[DPrbplot3, regionsplot, DisplayFunction → $DisplayFunction]

DUplot3 = ContourPlot[DU3, {g, 0.00001, 1 / 2 - 0.00001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity ,
   RegionFunction → Function[{g, c, z}, Min[1 / 2, cl] + 0.0001 > c > cd - 0.01],
   ContourStyle → {Thickness[0.01]} ];
Show[DUplot3, regionsplot, DisplayFunction → $DisplayFunction]

DWplot3 = ContourPlot[DW3, {g, 0.00001, 1 / 2 - 0.0001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity , RegionFunction →
    Function[{g, c, z}, Min[1 / 2, cl] > c > cd], ContourStyle → {Thickness[0.01]} ];
Show[DWplot3, regionsplot, DisplayFunction → $DisplayFunction]

DCarlton3 = Max[DPrb3, DU3] /. {a → as, pa → pal, pb → pbl};
DCarltonplot3 = ContourPlot[DCarlton3, {g, 0.00001, 1 / 2 - 0.0001}, {c, 0, 1},
   Contours → {0}, ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity ,
   RegionFunction → Function[{g, c, z}, Min[1 / 2, cl] + 0.001 > c > cd - 0.01],
   ContourStyle → {Thickness[0.01]} ];
Show[DCarltonplot3, regionsplot, DisplayFunction → $DisplayFunction]
```

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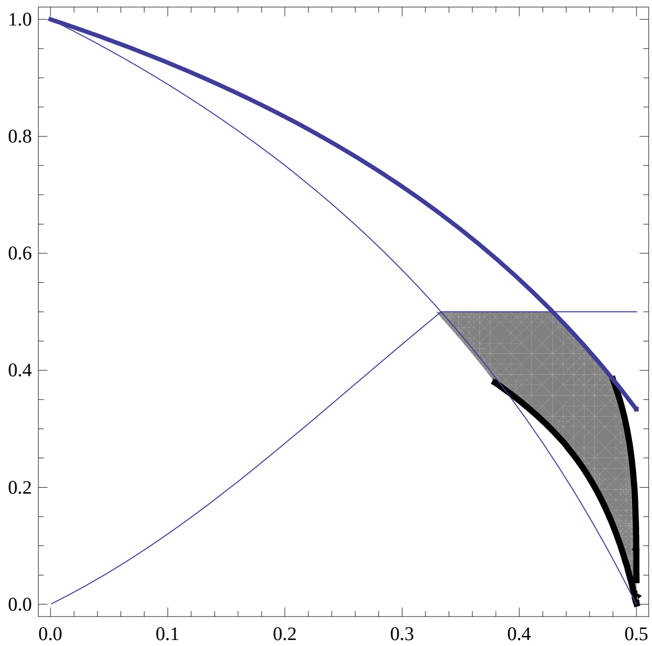
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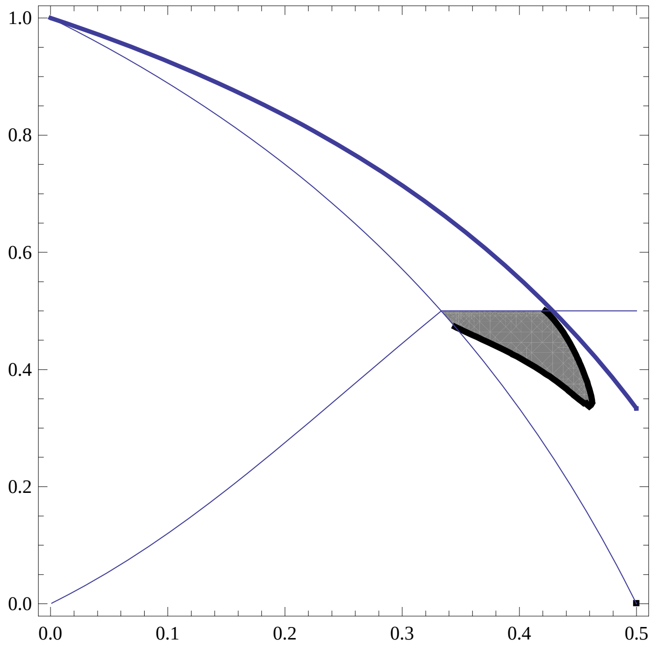
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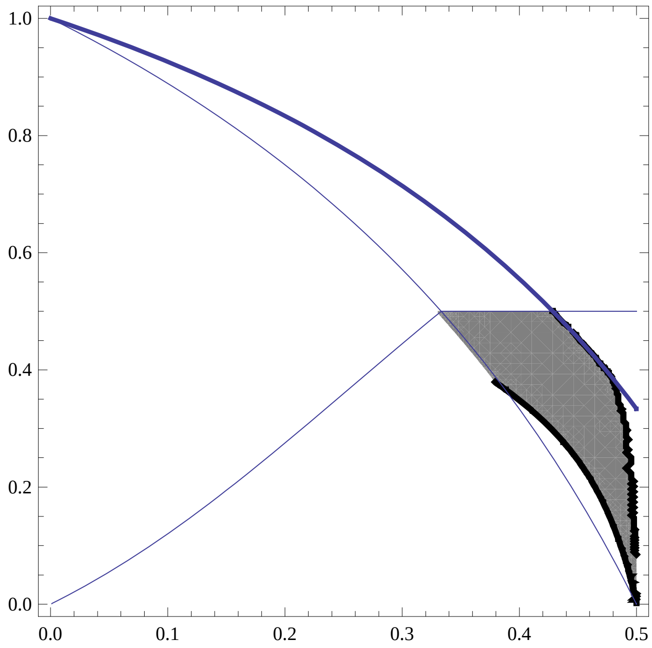
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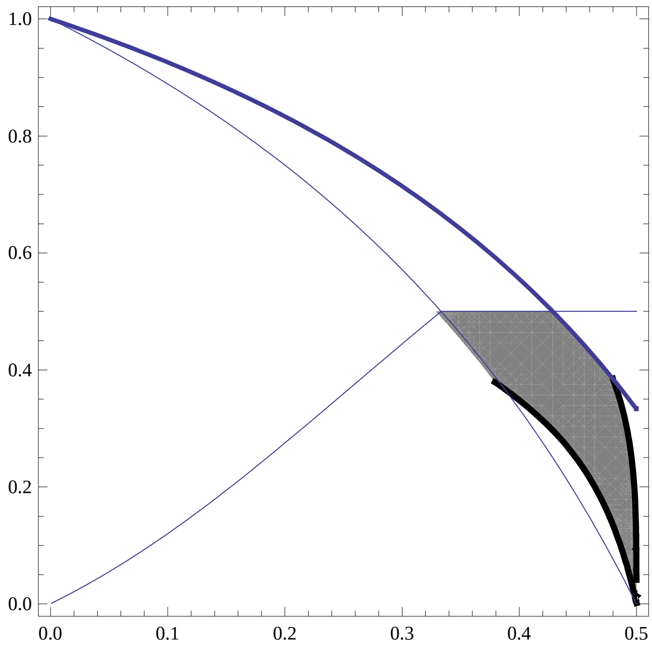
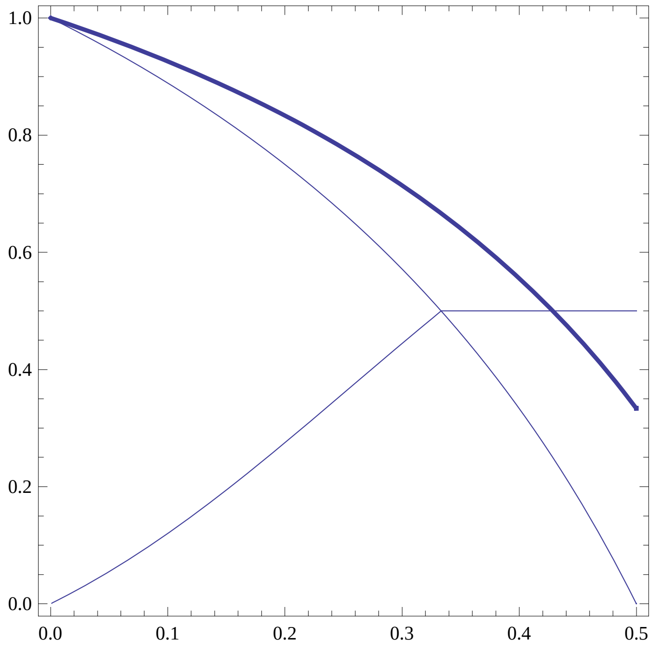
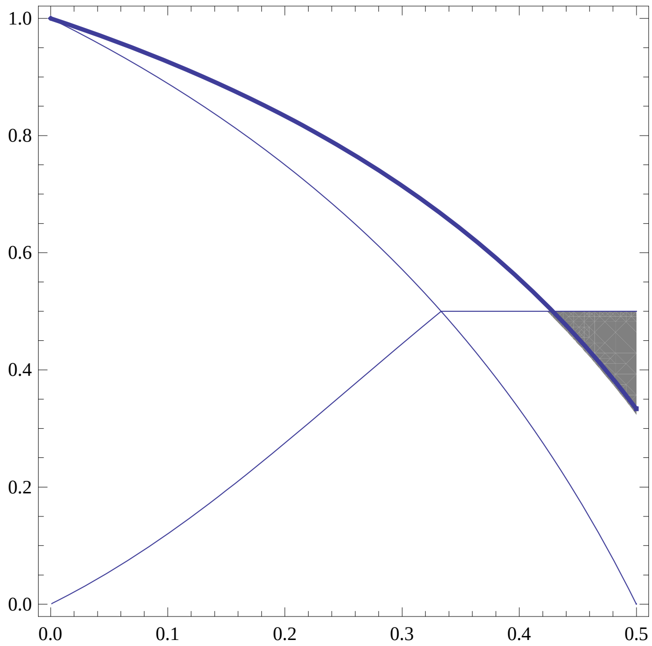
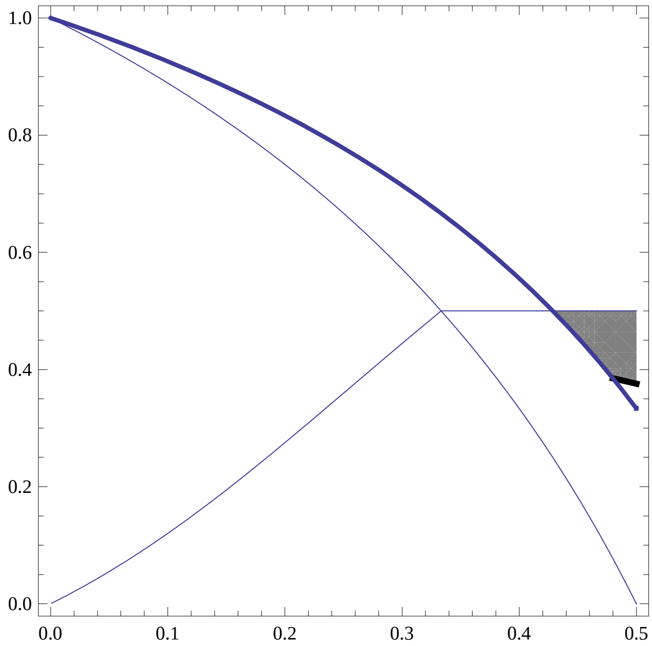
```
(*for some reason the numerical plot of DCarlton3 comes out with some issues,
however, inspection of DPrb3 and DU3 show that the Max of the
 two should have the same shape of DU3 since DPrb3 is always negative,
hence we directly define  the plot of Carlton figure in this case as that of DU3*)
DCarltonplot3 = DUplot3;
Show[DCarltonplot3, regionsplot, DisplayFunction → $DisplayFunction]
```

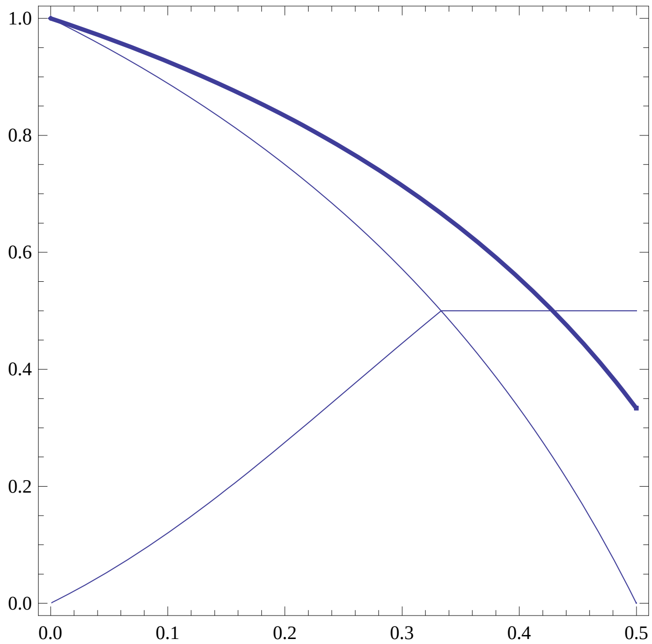



```
(*REGION 4*)

DPra4 = PraRLM - Pramonl /. {a → as, pal → palinear, pbl → pblinear};
DPrb4 = PrbRLM - 0 /. {a → as, pal → palinear, pbl → pblinear};
DU4 = URLM - Umonl /. {a → as, pal → palinear, pbl → pblinear};
DW4 = WRLM - Wmonl /. {a → as, pal → palinear, pbl → pblinear};



DPraplot4 = ContourPlot[DPra4, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity ,
   RegionFunction → Function[{g, c, z}, 1 / 2 > c > cl], ContourStyle → {Thickness[0.01]} ];
Show[DPraplot4, regionsplot, DisplayFunction → $DisplayFunction]

DPrbplot4 = ContourPlot[DPrb4, {g, 0, 1 / 2}, {c, 0, 1}, DisplayFunction → Identity,
   ColorFunction → (If[#1 < 0, White, Gray] &), RegionFunction →
    Function[{g, c, z}, 1 / 2 > c > cl - 0.01], ContourStyle → {Thickness[0.01]} ];
Show[DPrbplot4, regionsplot, DisplayFunction → $DisplayFunction]

DUplot4 = ContourPlot[DU4, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity,
   RegionFunction → Function[{g, c, z}, 1 / 2 > c > cl], ContourStyle → {Thickness[0.01]} ];
Show[DUplot4, regionsplot, DisplayFunction → $DisplayFunction]

DWplot4 = ContourPlot[DW4, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity,
   RegionFunction → Function[{g, c, z}, 1 / 2 > c > cl], ContourStyle → {Thickness[0.01]} ];
Show[DWplot4, regionsplot, DisplayFunction → $DisplayFunction]
```

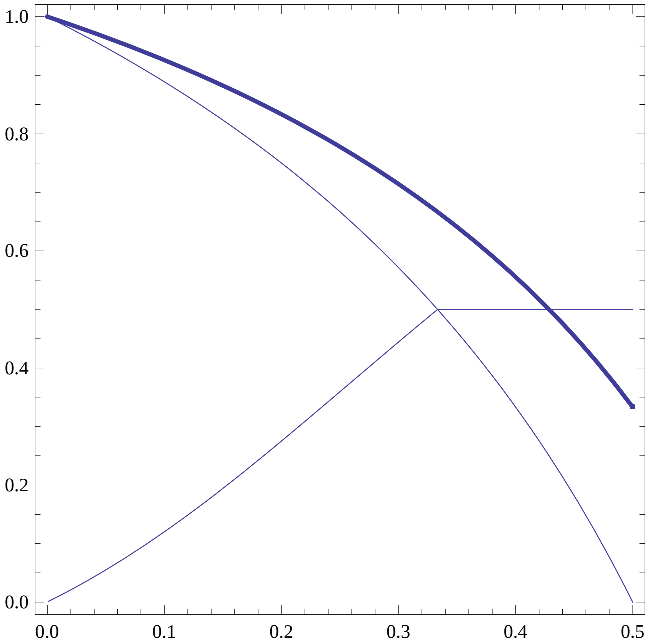

1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5

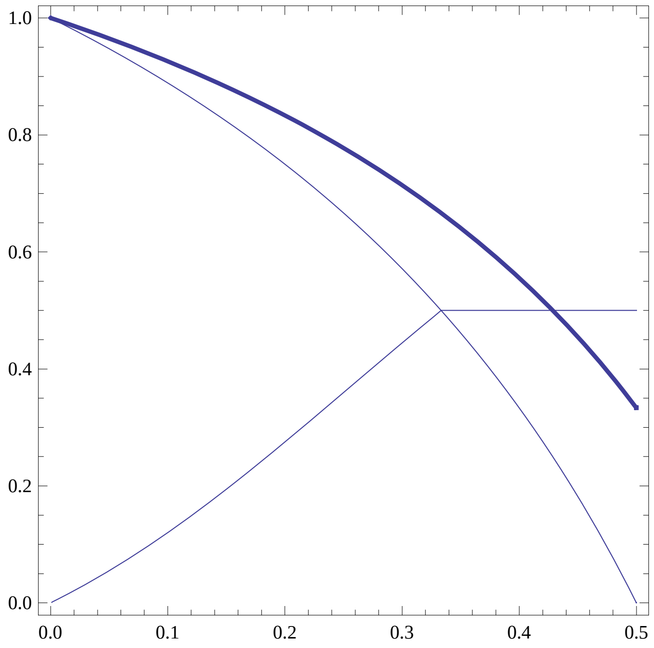

1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5

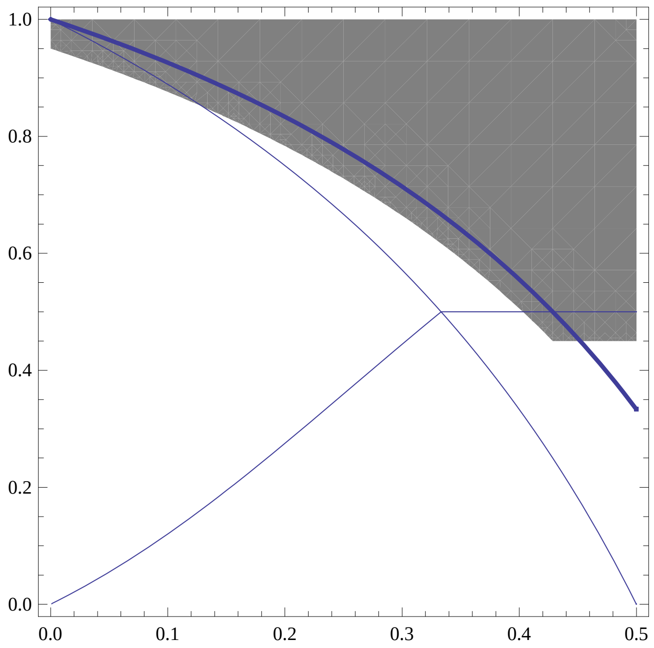

1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5

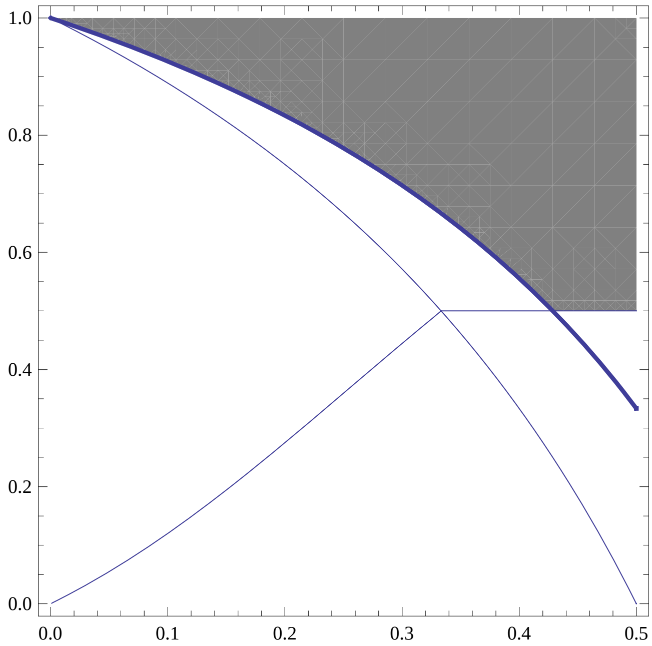
```
(*here since DPrb is always negative the Carlton test can be simply plotted with DU*)
DCarlton4 = DU4;
DCarltonplot4 = ContourPlot[DCarlton4, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, White] &), DisplayFunction → Identity ,
   RegionFunction → Function[{g, c, z}, 1 / 2 > c > cl], ContourStyle → {None} ];
Show[DCarltonplot4, regionsplot, DisplayFunction → $DisplayFunction]
```

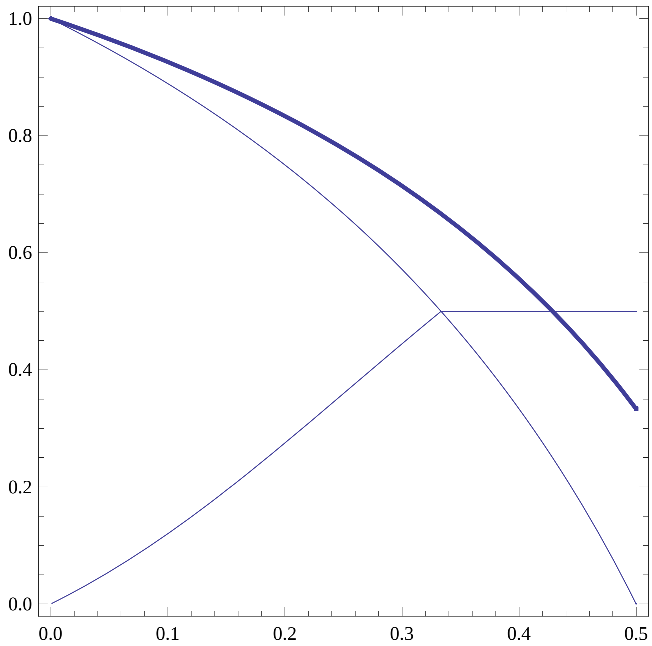


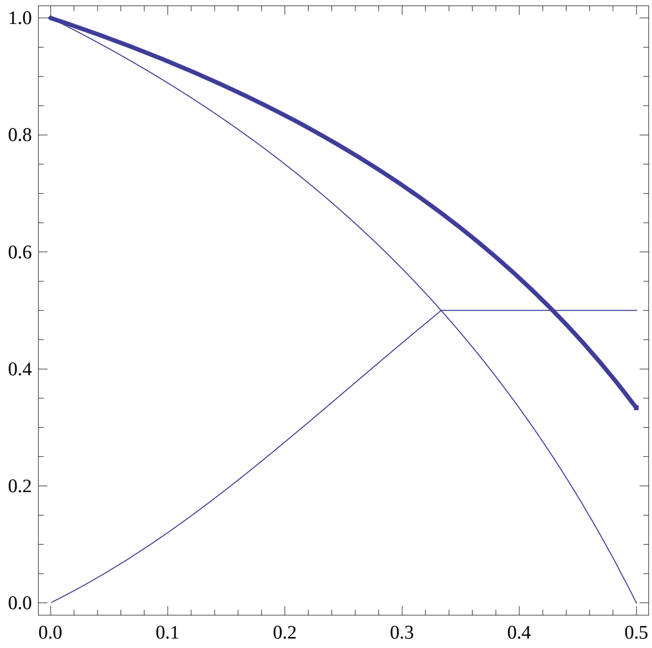
```
(*REGION 5*)

DPra5 = PraRM - Pramonl /. {a → as, pal → palinear, pbl → pblinear};
DPrb5 = PrbRM - 0 /. {a → as, pal → palinear, pbl → pblinear};
DU5 = URM - Umonl /. {a → as, pal → palinear, pbl → pblinear};
DW5 = WRM - Wmonl /. {a → as, pal → palinear, pbl → pblinear};

DPraplot5 = ContourPlot[DPra5, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity , RegionFunction →
    Function[{g, c, z}, c > Max[1 / 2, cl]], ContourStyle → {Thickness[0.01]} ];
Show[DPraplot5, regionsplot, DisplayFunction → $DisplayFunction]

DPrbplot5 = ContourPlot[DPrb5, {g, 0, 1 / 2}, {c, 0, 1}, DisplayFunction → Identity,
   ColorFunction → (If[#1 < 0, White, Gray] &), RegionFunction →
    Function[{g, c, z}, c > Max[1 / 2, cl] - 0.05], ContourStyle → {Thickness[0.01]} ];
Show[DPrbplot5, regionsplot, DisplayFunction → $DisplayFunction]

DUplot5 = ContourPlot[DU5, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity, RegionFunction →
    Function[{g, c, z}, c > Max[1 / 2, cl]], ContourStyle → {Thickness[0.01]} ];
Show[DUplot5, regionsplot, DisplayFunction → $DisplayFunction]

DWplot5 = ContourPlot[DW5, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity, RegionFunction →
    Function[{g, c, z}, c > Max[1 / 2, cl]], ContourStyle → {Thickness[0.01]} ];
Show[DWplot5, regionsplot, DisplayFunction → $DisplayFunction]

DCarlton5 = Max[DPrb5, DU5] /. {a → as, pa → pal, pb → pbl};
DCarltonplot5 = ContourPlot[DCarlton5, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity , RegionFunction →
    Function[{g, c, z}, c > Max[1 / 2, cl]], ContourStyle → {Thickness[0.01]} ];

(*however, since in any case B is not active here we put an empty region *)
(* DCarltonplot5=DPrbplot5;
Show[DCarltonplot5, regionsplot,DisplayFunction→$DisplayFunction]
*)

DCarltonplot5 = DU5;
DCarltonplot5 = ContourPlot[DCarltonplot5, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, White] &), DisplayFunction → Identity ,
   RegionFunction → Function[{g, c, z}, 1 / 2 > c > cl], ContourStyle → {None} ];
Show[DCarltonplot5, regionsplot, DisplayFunction → $DisplayFunction]
```


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5

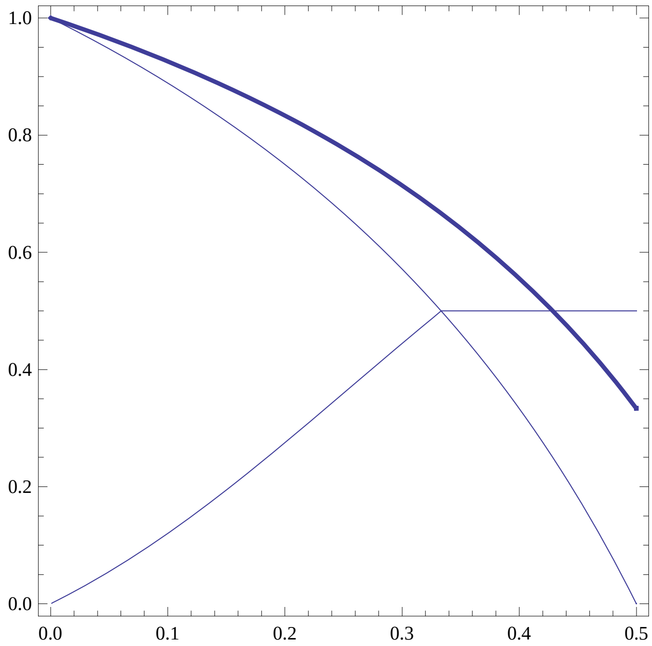

1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5

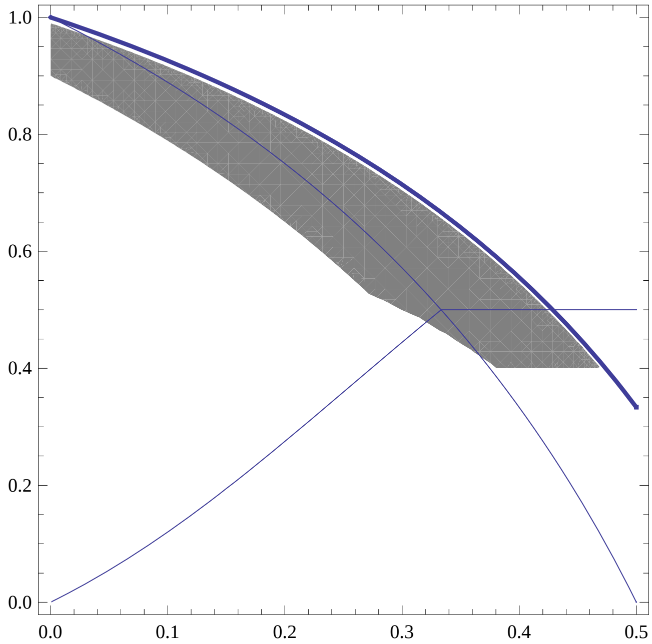

1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5

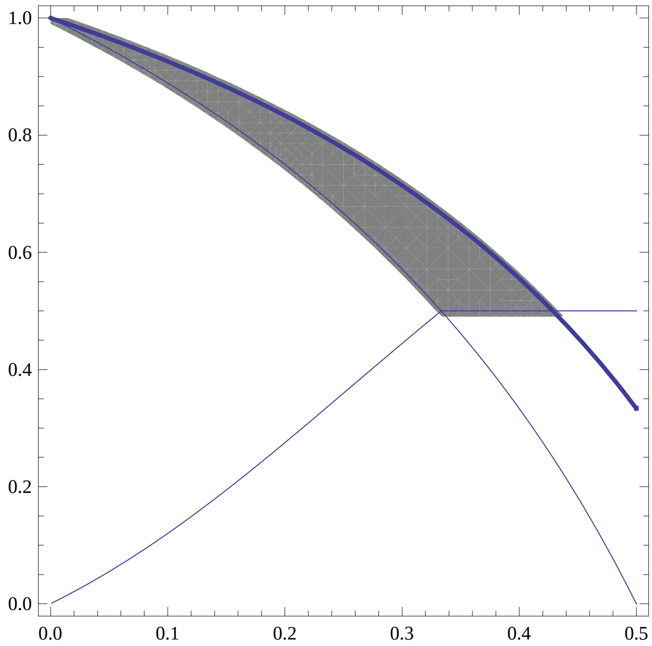

1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5

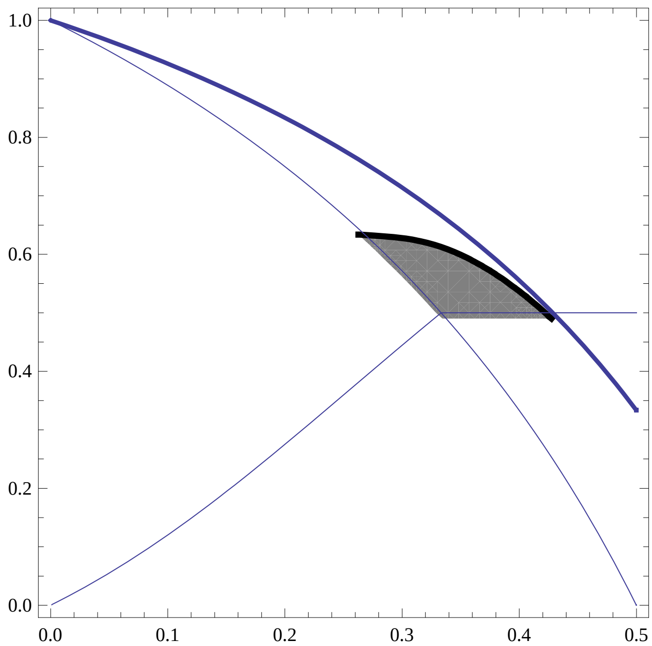

1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5

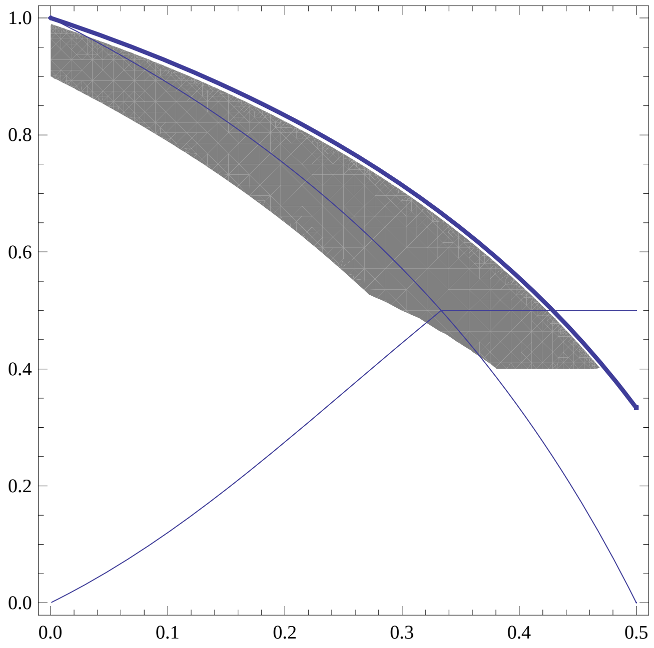
```
(*REGION 6*)

DPra6 = PraRM - Pradl /. {a → as, pal → palinear, pbl → pblinear};
DPrb6 = PrbRM - Prbdl /. {a → as, pal → palinear, pbl → pblinear};
DU6 = URM - Udl /. {a → as, pal → palinear, pbl → pblinear};
DW6 = WRM - Wdl /. {a → as, pal → palinear, pbl → pblinear};

DPraplot6 = ContourPlot[DPra6, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity, RegionFunction →
    Function[{g, c, z}, cl > c > Max[1 / 2, cd]], ContourStyle → {Thickness[0.01]} ];
Show[DPraplot6, regionsplot, DisplayFunction → $DisplayFunction]

(*DPrbplot6=ContourPlot[DPrb6,{g,0,1/2},{c,0,1},Contours→{0},
   ColorFunction→(If[#1>0,White,Gray]&),DisplayFunction→Identity ,
   RegionFunction→Function[{g,c,z},cmaxlinear+0.01>c>Max[1/2,cd]-0.1],
   ContourStyle→{Thickness[0.01]} ];*)

DPrbplot6 = ContourPlot[DPrb6, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity , RegionFunction →
    Function[{g, c, z}, cl - 0.01 > c > Max[1 / 2, cd] - 0.1], ContourStyle → {None} ];
Show[DPrbplot6, regionsplot, DisplayFunction → $DisplayFunction]

DUplot6 = ContourPlot[DU6, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity,
   RegionFunction → Function[{g, c, z}, cl + 0.01 > c > Max[1 / 2, cd] - 0.01],
   ContourStyle → {Thickness[0.01]} ];
Show[DUplot6, regionsplot, DisplayFunction → $DisplayFunction]

DWplot6 = ContourPlot[DW6, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity , RegionFunction →
    Function[{g, c, z}, cl > c > Max[1 / 2, cd] - 0.01], ContourStyle → {Thickness[0.01]} ];
Show[DWplot6, regionsplot, DisplayFunction → $DisplayFunction]

DCarlton6 = Max[DPrb6, DU6] /. {a → as, pa → pal, pb → pbl};
DCarltonplot6 = ContourPlot[DCarlton6, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity ,
   RegionFunction → Function[{g, c, z}, cl - 0.01 > c > Max[1 / 2, cd] - 0.1],
   ContourStyle → {Thickness[0.00000000001]} ];
Show[DCarltonplot6, regionsplot, DisplayFunction → $DisplayFunction]
```


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5

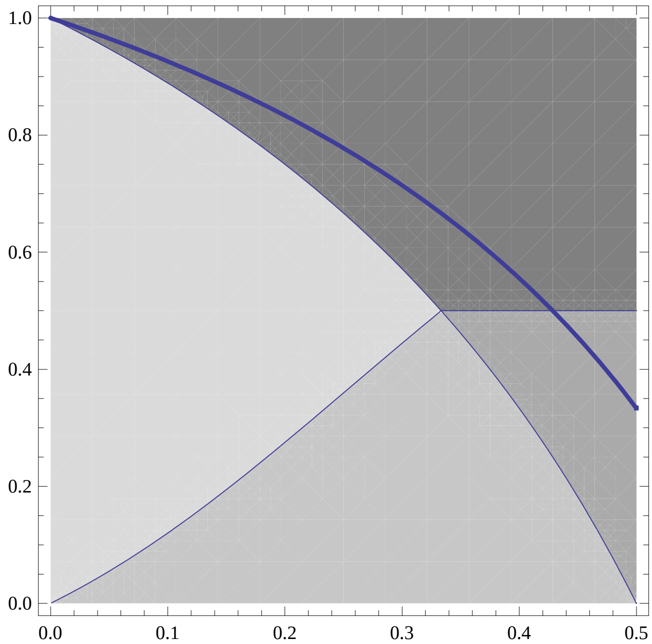

1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5

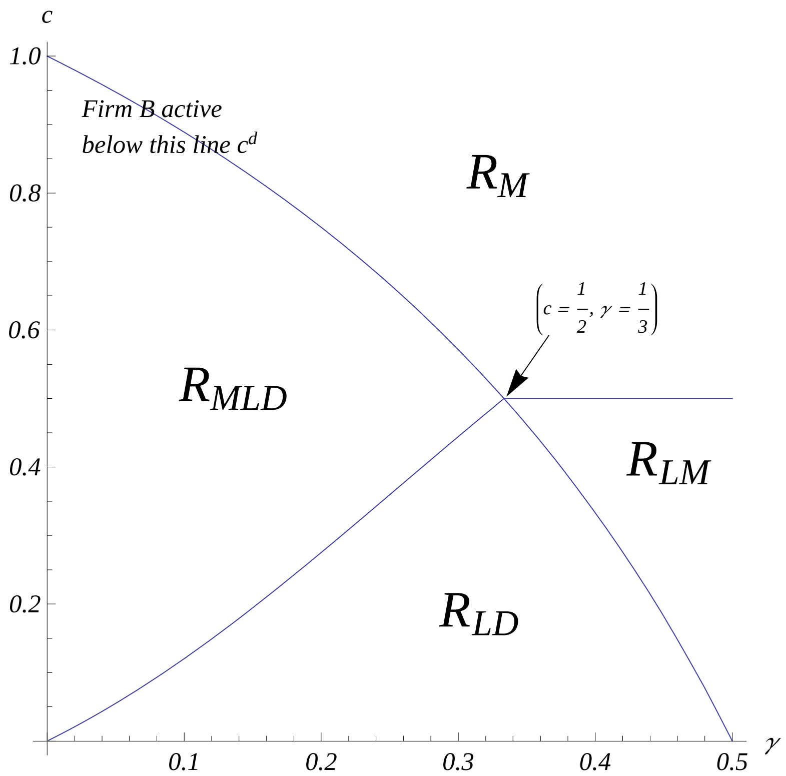


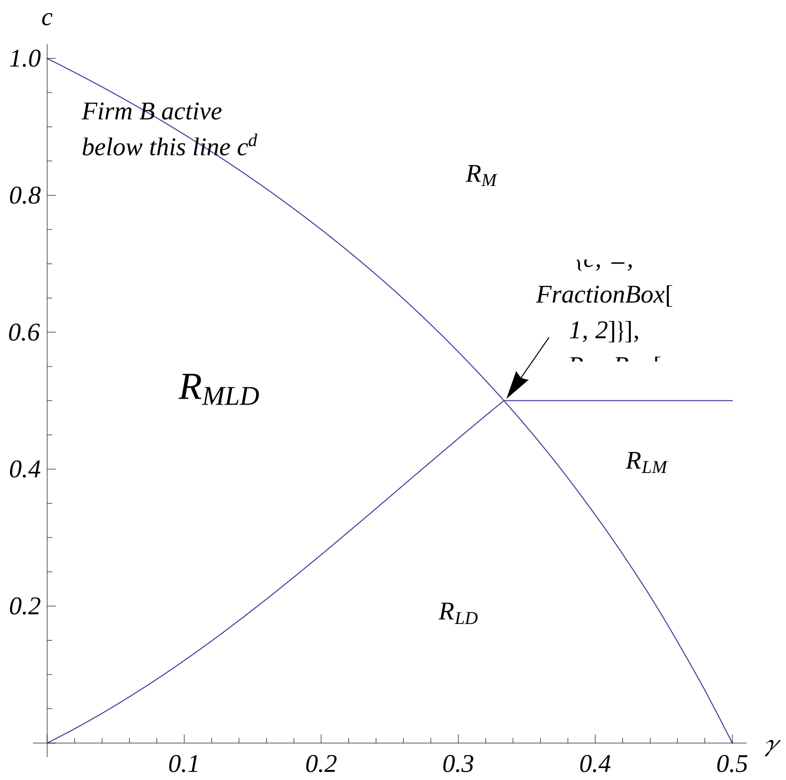

```
(*Plotting all regions for comparison non-
linear versus linear (Figure 2-7 in the paper)*)
```

```
RM = ContourPlot[1, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity , RegionFunction →
    Function[{g, c, z}, c > Max[cd, 1 / 2]], ContourStyle → {Thickness[0.01]} ];

RLM = ContourPlot[1, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Lighter[Gray]] &), DisplayFunction → Identity ,
   RegionFunction → Function[{g, c, z}, 1 / 2 > c > cd], ContourStyle → {Thickness[0.01]} ];

RLD = ContourPlot[1, {g, 0, 1 / 2 - 0.000001}, {c, 0, 1},
   Contours → {0}, ColorFunction → (If[#1 > 0, White, Lighter[Lighter[Gray]]] &),
   DisplayFunction → Identity , RegionFunction → Function[{g, c, z}, Min[cd, ctilde] > c],
   ContourStyle → {Thickness[0.01]} ];

RMLD = ContourPlot[1, {g, 0, 1 / 2 - 0.000001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Lighter[Lighter[Lighter[Gray]]]] &),
   DisplayFunction → Identity , RegionFunction → Function[{g, c, z}, cd > c > ctilde],
   ContourStyle → {Thickness[0.01]} ];
Show[RM, RLM, RLD, RMLD, regionsplot, DisplayFunction → $DisplayFunction]
```

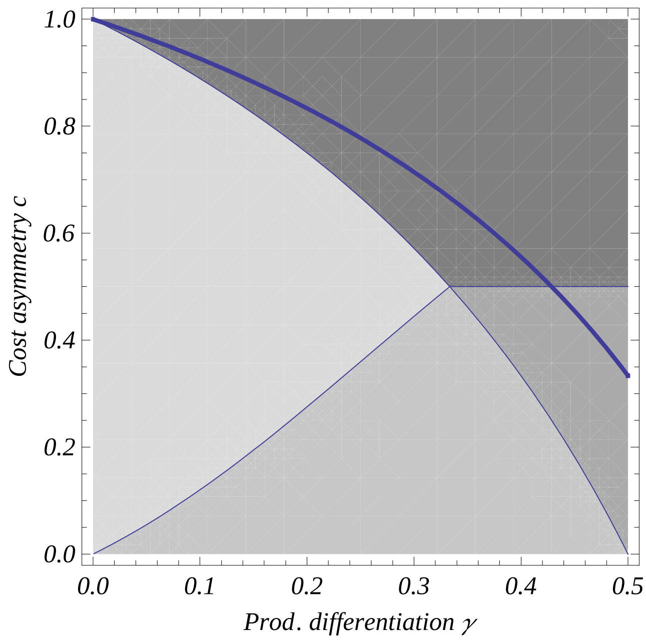


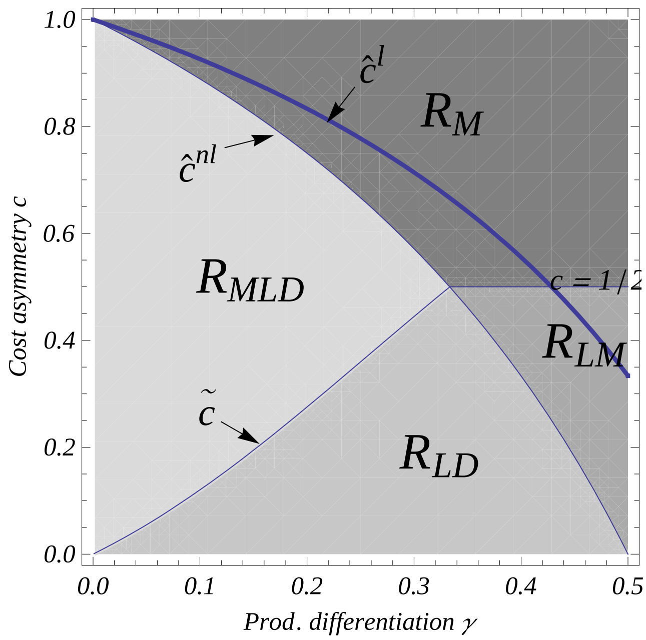

```
(*Plotting the four regions with non linear pricing*)
cdplot = Plot[{cd}, {g, 0, 1 / 2}, AspectRatio → 1];
ctildeplot = Plot[{ctilde /. a → as}, {g, 0.001, 1 / 3}];
clplot = Plot[{cl}, {g, 0, 1 / 2}, PlotStyle → {Thick}];
clplotthick = Plot[{cl}, {g, 0, 1 / 2}, AspectRatio → 1,
   PlotRange → {0, 1}, PlotStyle → {Thickness[0.015], Black}];
constplot = Plot[{1 / 2}, {g, 1 / 3, 1 / 2}];
```


c
1.0
0.8
0.6
0.4
0.2
Firm B active
below this line c^d
R_M
$\left(c = \frac{1}{2}, \gamma = \frac{1}{3}\right)$
R_{MLD}
R_{LM}
R_{LD}
0.1
0.2
0.3
0.4
0.5
γ

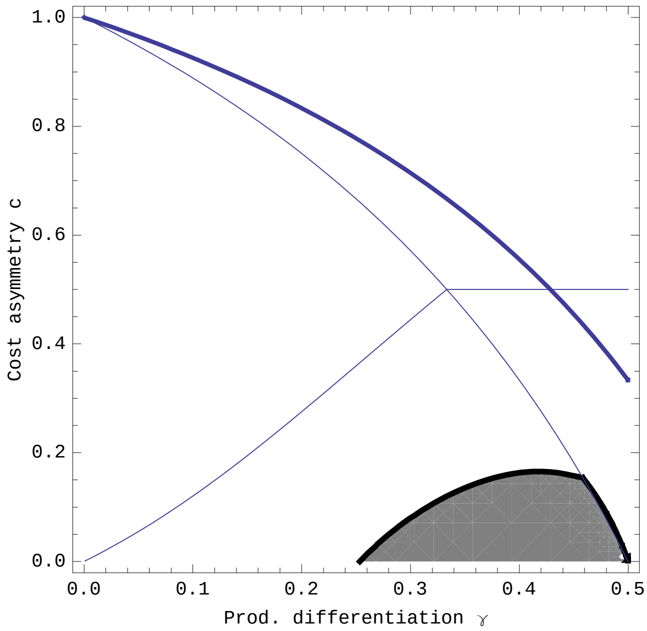

c
1.0
0.8
0.6
0.4
0.2
Firm B active
below this line c^d
R_M
FractionBox[
1, 2]}],
R_{MLD}
R_{LM}
R_{LD}
0.1
0.2
0.3
0.4
0.5
γ

```
(*Figure 2 in the paper*)

RM = ContourPlot[1, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Gray] &), DisplayFunction → Identity , RegionFunction →
    Function[{g, c, z}, c > Max[cd, 1 / 2]], ContourStyle → {Thickness[0.01]} ];

RLM = ContourPlot[1, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Lighter[Gray]] &), DisplayFunction → Identity ,
   RegionFunction → Function[{g, c, z}, 1 / 2 > c > cd], ContourStyle → {Thickness[0.01]} ];

RLD = ContourPlot[1, {g, 0, 1 / 2 - 0.000001}, {c, 0, 1},
   Contours → {0}, ColorFunction → (If[#1 > 0, White, Lighter[Lighter[Gray]]] &),
   DisplayFunction → Identity , RegionFunction → Function[{g, c, z}, Min[cd, ctilde] > c],
   ContourStyle → {Thickness[0.01]} ];

RMLD = ContourPlot[1, {g, 0, 1 / 2 - 0.000001}, {c, 0, 1}, Contours → {0},
   ColorFunction → (If[#1 > 0, White, Lighter[Lighter[Lighter[Gray]]]] &),
   DisplayFunction → Identity , RegionFunction → Function[{g, c, z}, cd > c > ctilde],
   ContourStyle → {Thickness[0.01]} ];
Show[RM, RLM, RLD, RMLD, regionsplot, PlotRange → {0, 1},
 TextStyle → {FontSlant → "Italic", FontSize → 12}, AxesLabel → {"γ", "c"},
 Frame → True, FrameLabel → {"Prod. differentiation γ", "Cost asymmetry c"}]
```

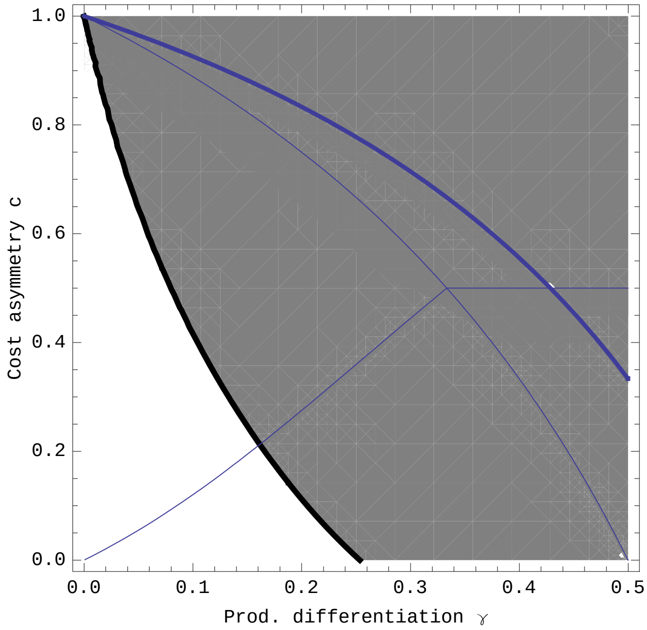




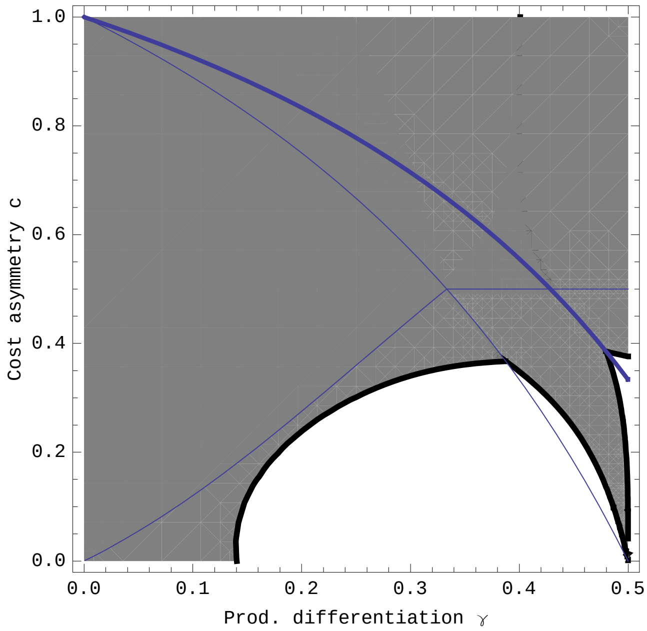

```
Show[ DPraplot1, DPraplot2, DPraplot3, DPraplot4, DPraplot5, DPraplot6, regionsplot,
 FrameLabel → {"Prod. differentiation γ", "Cost asymmetry c"}, FormatType → StandardForm]
(*figure 3 in the paper*)
```



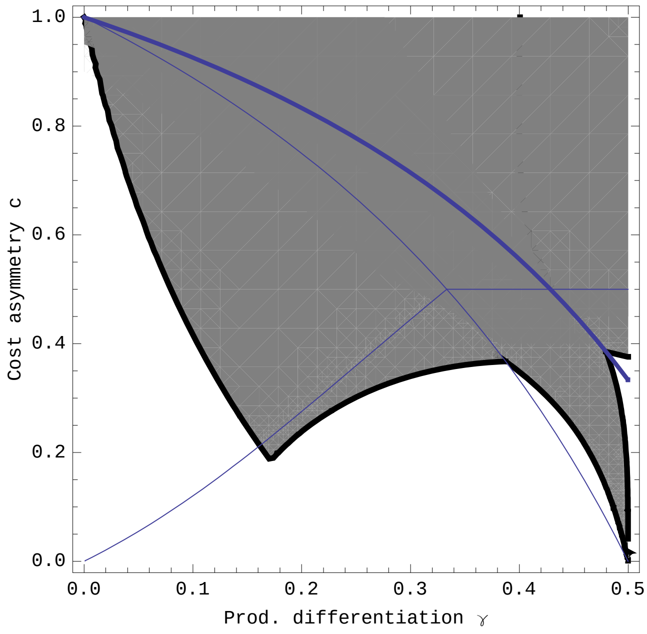
```
Show[DPrbplot6, DPrbplot5, DPrbplot1, DPrbplot2, DPrbplot3, DPrbplot4, regionsplot,
 FrameLabel → {"Prod. differentiation γ", "Cost asymmetry c"}, FormatType → StandardForm]
(*figure 4 in the paper*)
```



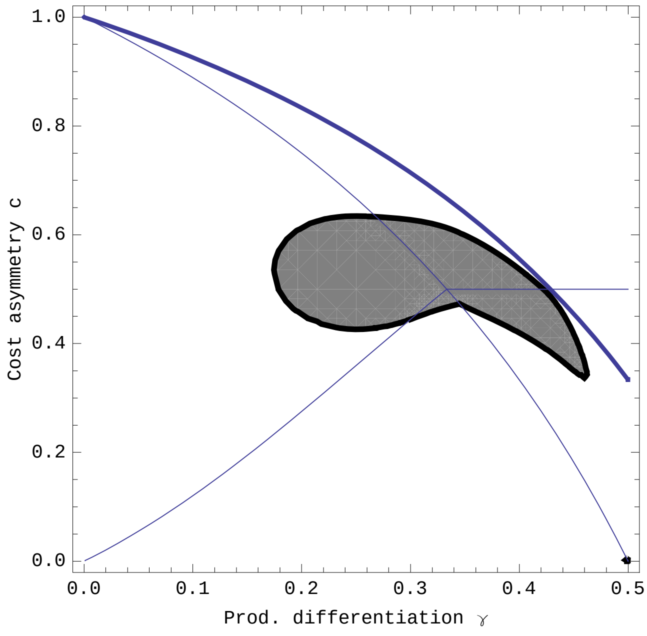

```
Show[DUplot1, DUplot3, DUplot4, DUplot5, DUplot2, DUplot6, regionsplot,
 FrameLabel → {"Prod. differentiation γ", "Cost asymmetry c"}, FormatType → StandardForm]
(*figure 5 in the paper*)
```



```
Show[DCarltonplot6, DCarltonplot1, DCarltonplot5,
 DCarltonplot3, DCarltonplot2, DCarltonplot4, regionsplot,
 FrameLabel → {"Prod. differentiation γ", "Cost asymmetry c"}, FormatType → StandardForm]
(*figure 6 in the paper*)
```

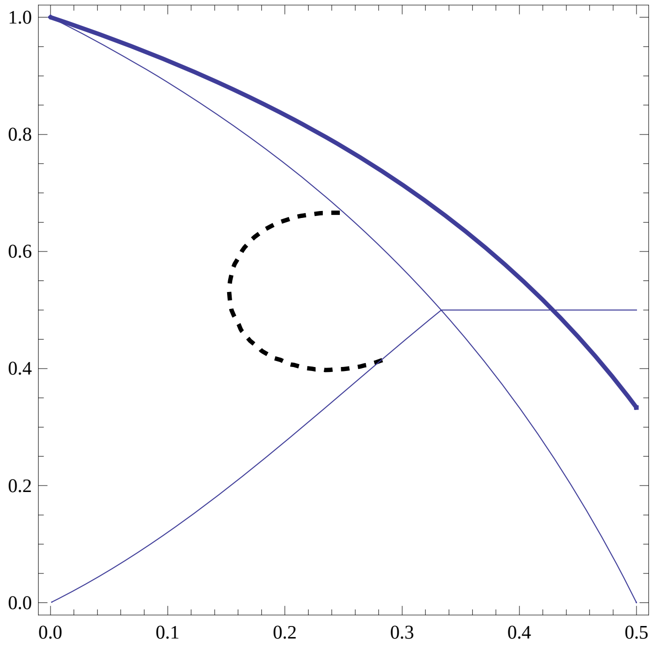



```
Show[DWplot1, DWplot2, DWplot3, DWplot4, DWplot5, DWplot6, regionsplot,
 FrameLabel → {"Prod. differentiation γ", "Cost asymmetry c"}, FormatType → StandardForm]
(*figure 7 in the paper*)
```

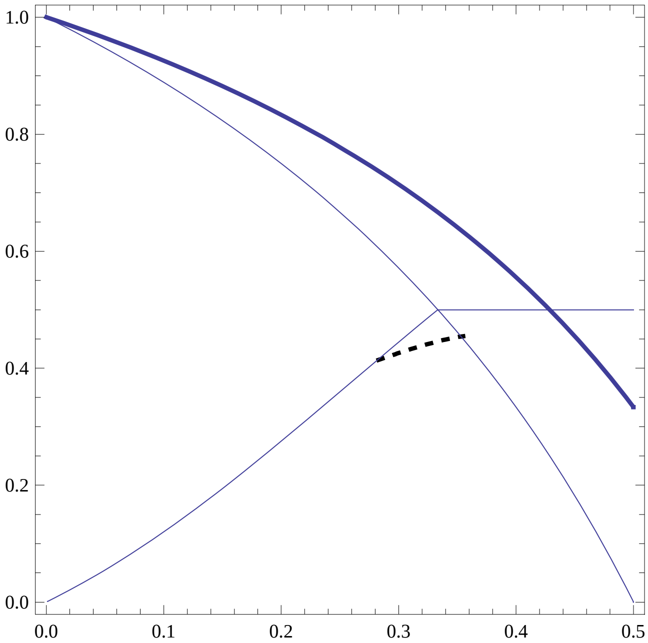



```
(* COMPARISONS: MIX-SELECTIVE BAN VERSUS NON-LINEAR *)
```

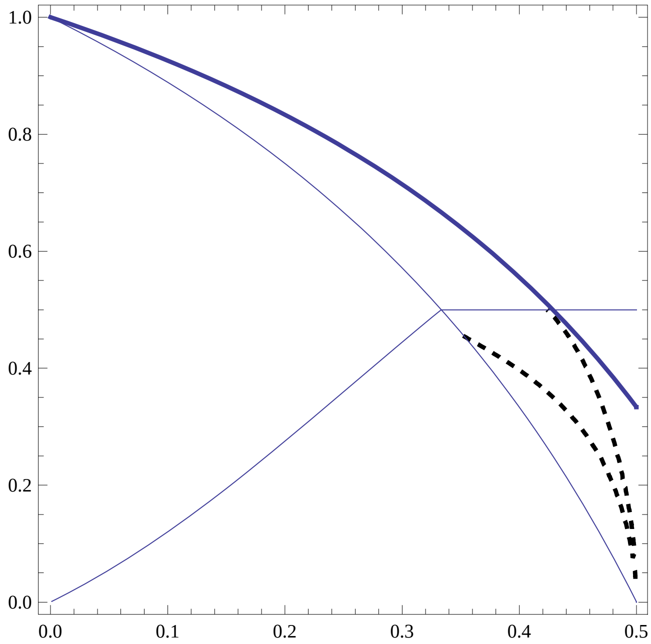
```
(*REGION 1*)

DW1mix = WRMLD - Wdmix /. {a → as, pal → palmix, b1 → b1mix, b2 → b2mix};

DWplot1mix = ContourPlot[DW1mix == 0, {g, 0.0001, 1 / 2 - 0.0001}, {c, 0, 1}, Contours → {0},
   DisplayFunction → Identity , RegionFunction → Function[{g, c, z}, cd > c > ctilde],
   ContourStyle → {Thickness[0.007], Dashed}];
Show[DWplot1mix, regionsplot, DisplayFunction → $DisplayFunction]
```



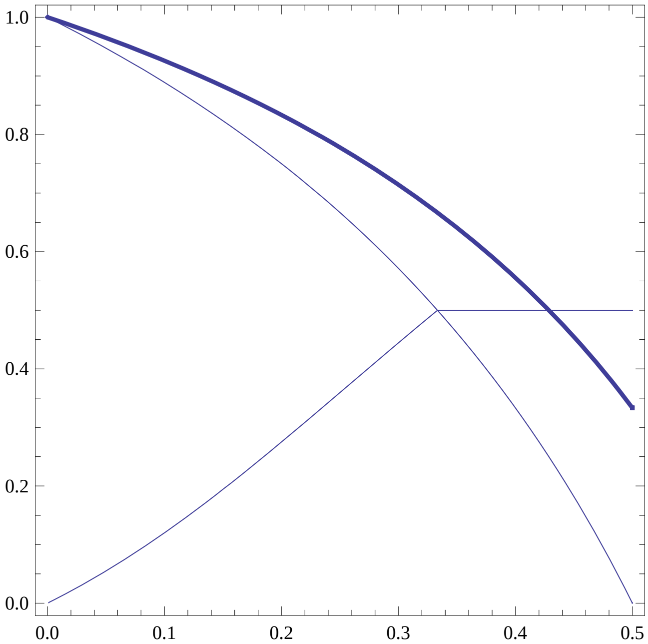
```
(*REGION 2*)

DW2mix = WRLD - Wdmix /. {a → as, pal → palmix, b1 → b1mix, b2 → b2mix};

DWplot2mix = ContourPlot[DW2mix == 0, {g, 0.001, 1 / 2 - 0.001}, {c, 0, 1}, Contours → {0},
   DisplayFunction → Identity , RegionFunction → Function[{g, c, z}, Min[cd, ctilde] > c],
   ContourStyle → {Thickness[0.007], Dashed}];
Show[DWplot2mix, regionsplot, DisplayFunction → $DisplayFunction]
```



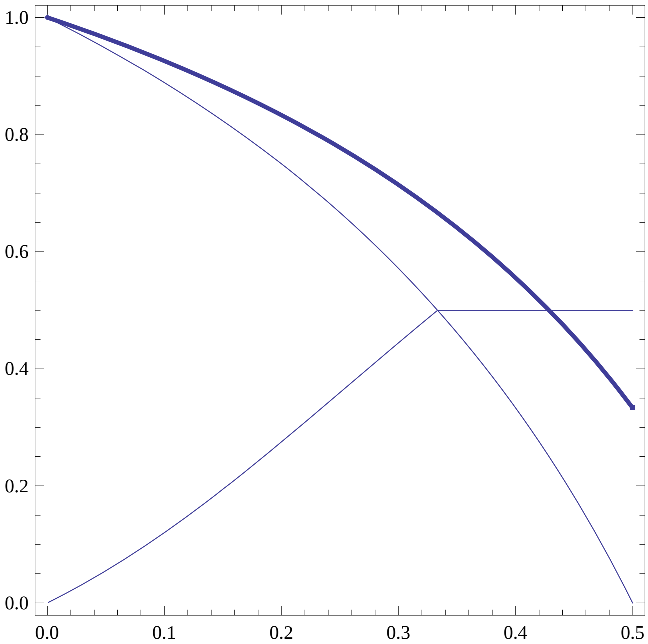
```
(*REGION 3*)

DW3mix = WRLM - Wdmix /. {a → as, pal → palmix, b1 → b1mix, b2 → b2mix};

DWplot3mix = ContourPlot[DW3mix == 0, {g, 0, 1 / 2 - 0.001}, {c, 0, 1}, Contours → {0},
   DisplayFunction → Identity, RegionFunction → Function[{g, c, z}, Min[1 / 2, cl] > c > cd],
   ContourStyle → {Thickness[0.007], Dashed} ];
Show[DWplot3mix, regionsplot, DisplayFunction → $DisplayFunction]
```



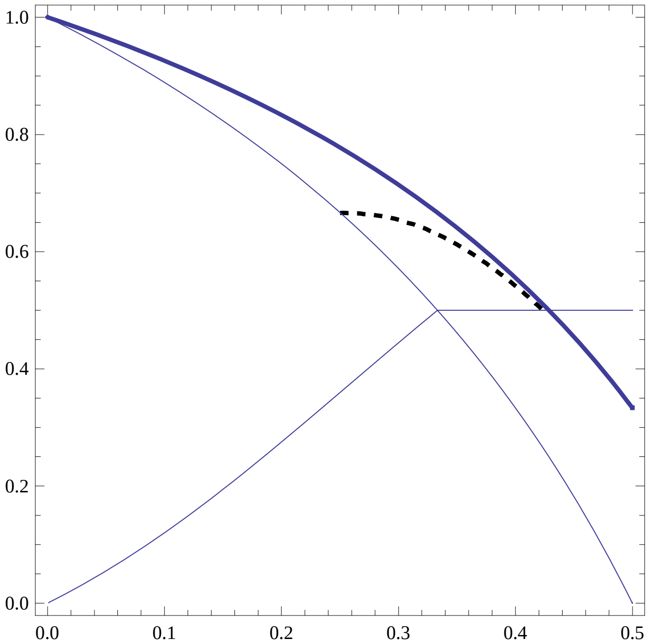
```
(*REGION 4*)

DW4mix = WRLM - Wmonmix /. {a → as, pal → palmix, b1 → b1mix, b2 → b2mix};

DWplot4mix = ContourPlot[DW4mix == 0, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   DisplayFunction → Identity, ContourStyle → {Thickness[0.007], Dashed} ,
   RegionFunction → Function[{g, c, z}, 1 / 2 > c > cl]];
Show[DWplot4mix, regionsplot, DisplayFunction → $DisplayFunction]
```



```
(*REGION 5*)

DW5mix = WRM - Wmonmix /. {a → as, pal → palmix, b1 → b1mix, b2 → b2mix};

DWplot5mix = ContourPlot[DW5mix == 0, {g, 0, 1/2}, {c, 0, 1}, Contours → {0},
   DisplayFunction → Identity, ContourStyle → {Thickness[0.007], Dashed},
   RegionFunction → Function[{g, c, z}, c > Max[1/2, cl]] ];
Show[DWplot5mix, regionsplot, DisplayFunction → $DisplayFunction]
```



```
(*REGION 6*)

DW6mix = WRM - Wdmix /. {a → as, pal → palmix, b1 → b1mix, b2 → b2mix};

DWplot6mix = ContourPlot[DW6mix == 0, {g, 0, 1 / 2}, {c, 0, 1}, Contours → {0},
   DisplayFunction → Identity , ContourStyle → {Thickness[0.007], Dashed},
   RegionFunction → Function[{g, c, z}, cl > c > Max[1 / 2, cd]]];
Show[DWplot6mix, regionsplot, DisplayFunction → $DisplayFunction]
```




```
        (*Plotting all regions for comparison non-linear versus mix (only welfare),
figure 8 in the paper*)

Show[DWplot1, DWplot2, DWplot3, DWplot4, DWplot5, DWplot6, DWplot1mix,
 DWplot2mix, DWplot3mix, DWplot4mix, DWplot5mix, DWplot6mix, regionsplot,
 FrameLabel → {"Prod. differentiation γ", "Cost asymmetry c"}, FormatType → StandardForm]
(*figure 8 in the paper*)
```